Hadronic decay width from finite-volume energy spectrum in lattice QCD

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INTRODUCTION

Why we need to study QCD on the lattice?

Running coupling constant \( Q \sim \) momentum transferred, \( \Lambda \sim \) QCD scale parameter:

\[
\alpha_s(Q^2) = \frac{12\pi}{(33 - 2N_f) \log (Q^2/\Lambda^2)}
\]

- \( \Lambda \sim \) hadronic mass
- \( Q^2 \gg \Lambda^2 \) perturbative regime
- \( Q^2 \geq \Lambda^2 \) non-perturbative (NP) regime

We need a NP method → Lattice formulation of QCD by K.Wilson 1974

Why we need to study QCD in an Euclidean Space?

The Green functions characterize the quantum field theory

In the Euclidean space \( G(x_1, x_2, \ldots, x_l) \):

\[
\int D\phi \phi(x_1)\phi(x_2)\ldots\phi(x_l) e^{-S_E^{(eucl.)}[\phi]} \frac{1}{\int D\phi e^{-S_E^{(eucl.)}[\phi]}}
\]

Equivalence between quantum field theory and statistical mechanics

This opens the way to the study of NP phenomena using NUMERICAL METHODS
Importance Sampling: we generate a sequence of configurations with probability distribution given by a Boltzmann factor: 

\[ P(\{U\}_i) \propto e^{-S_E(\{U\}_i)} \]

\[ \langle O \rangle = \frac{\int D\phi O[\phi] e^{-S[\phi]}}{\int D\phi e^{-S[\phi]}} \Rightarrow \langle O \rangle \approx \frac{1}{N} \sum_{i=1}^{N} O(\{U\}_i) \quad \Delta \langle O \rangle \propto \frac{1}{\sqrt{N}} \]

Finite lattice spacing “a”

Finite lattice Volume \( L^3 \times T \) (EUCLIDEAN SPACE)

\[ x = na, \quad t = ma \]

\[ \phi(x, t) \rightarrow \phi(n, m) \]
PARTICLE MASSES

How we can determine the mass of particles on the lattice?

**Stable particles** (Partial Fourier Transform):

\[
C(t, \vec{p}) = \int \frac{d^3 \vec{x}}{(2\pi)^3} e^{-\vec{p} \cdot \vec{x}} G(\vec{x}, t; 0, 0) = \int \frac{d\omega}{2\pi} e^{i\omega t} \tilde{G}(\vec{p}, \omega) \implies C(t, \vec{p} = 0) \propto e^{-mt}
\]

**Unstable particles** \((\phi \rightarrow 2\pi, m_\phi > 2m_\pi)\)

- \(C(t, \vec{p} = 0) \propto e^{-2m_\pi t}\)
- resonances do not correspond to isolated energy levels
  (width related to a complex pole in the prop)
- on the lattice the energy spectrum is always REAL \(\implies\) Isolated Levels!

What we can see in the presence of a resonance on the lattice?

A rearrangement of the energy levels takes place (Avoid Crossing Level) related to a mixing between \(\phi\) and \(2\pi\)
Consider a system of two non-interacting particles (opposite momentum) in a box. Two-particle energy spectrum ($V = L^3$): 

$$E = 2\sqrt{m_\pi^2 + p^2}, \text{ where } p_i = \frac{2\pi}{L_i} n \text{ and } n = 0, \cdots, L_i - 1$$

Asymmetric box $\bar{L}^2 \times L$:

$$E_{(1,0,0)} = 2\sqrt{m_\pi^2 + \left(\frac{2\pi}{L}\right)^2}$$

$$\bar{n} = (0, 0, 0), (1, 0, 0), (1, 1, 0), (1, 1, 1), (2, 0, 0)$$
Introducing a new particle $\phi$ (at rest) interacting with $\pi$

$2m_\pi < m_\phi < 4m_\pi \Rightarrow$ Elastic two-particle scattering

$m_\pi = 0.468$

$E_\pi$ vs. $L$

- $4m_\pi$
- $m_\phi$
- $2m_\pi$
Two-particle energy spectrum: \( E = 2\sqrt{m_\pi^2 + p^2} \), but now \( p_i \neq \frac{2\pi}{L_i} n \)

Avoid Level Crossings (ALC)

Luscher’s formula:

\[
\tan \delta_{l=0}(p) = \frac{\pi^{3/2} \kappa}{Z_{00}(1; q^2)}
\]

\[
Z_{00}(s; q^2) = \frac{1}{\sqrt{4\pi}} \sum_{\vec{n} \in Z^3} (\vec{n}^2 - q^2)^{-s}
\]

- \( q = \frac{pL}{2\pi} \)

- \( l \) angular momentum of scattering channel

It relates the infinite-volume elastic phase shift \( \delta_0 \) to the finite-volume two-particle energy spectrum

\( \delta_0 \Rightarrow \) the resonance parameters
Two-particle energy spectrum: \( E = 2\sqrt{m_{\pi}^2 + p^2} \), but now \( p_i \neq \frac{2\pi}{L_i} n \)

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\[
Z_{00}(s; q^2) = \frac{1}{\sqrt{4\pi}} \sum_{\vec{n} \in \mathbb{Z}^3} (\vec{n}^2 - q^2)^{-s}
\]

- \( q = \frac{p L}{2\pi} \)
- \( l \) angular momentum of scattering channel

It relates the infinite-volume elastic phase shift \( \delta_0 \) to the finite-volume two-particle energy spectrum

If the width is large (\( \Delta \)-resonance) the ALC is washed out!

They define a probability distribution $W(p) \Rightarrow \lim_{\Delta L \to 0} \frac{\Delta L}{\Delta p} = \frac{1}{p'(L)}$

Prescription:
- Choose the first $N$ levels
- Slice the interval $[L_1, L_2]$ into equal parts $\Delta L$
- Introduce momentum bins $\Delta p$ in the interval $[p_1, p_2]$
- For each $L$ identify the $i$-bin $p_i < p^n(L) < p_i + \Delta p$, $n = 1, \cdots, N$ in which $p^n(L)$ is contained
- Plot the probability distribution $W(p)$ for $p_1 < p < p_2$
Differentiating the Luscher formula we have:

\[ W(p) = C \sum_{n=1}^{N} \left( \frac{L_n(p)}{p} + \frac{2\pi \delta'(p)}{p\phi'(\kappa_n(p))} \right) \]

\[ W_0(p) = C \sum_{n=1}^{N} \left( \frac{L_n(p)}{p} \right) \text{ is just a background} \]

Assuming a smooth dependence on the momentum \( p \)

\[ W(p) - W_0(p) \propto \frac{1}{p} \left( \frac{\delta(p)}{p} - \delta'(p) \right) \]

It follows the Breit-Wigner form (for the scattering cross section) with SAME WIDTH

Using experimental phase shifts as input they produced synthetic data and showed it is possible to determine the parameters of a resonance even if the ALCs are washed out
\[ S = \int d^4 x \left\{ \beta \text{Tr} \left( \partial_\mu \Sigma_x \partial_\mu \Sigma^\dagger_x \right) - m_\pi^2 \beta \text{Tr} \left( \Sigma_x + \Sigma^\dagger_x \right) + \frac{1}{2} \phi_x (-\partial_\mu \partial_\mu + m_\phi^2) \phi_x 
+ \lambda \phi_x \beta \text{Tr} \left( \partial_\mu \Sigma_x \partial_\mu \Sigma^\dagger_x + \nu \right) \right\} \]

where \( \Sigma_x = \exp \left( i \frac{\pi_x \sigma_i}{f} \right) \)

This Action is invariant under \( \Sigma \rightarrow U_R \Sigma U_L^\dagger \), where \( U_L, U_R \in SU(2) \)

Symmetry breaking term \( m_\pi^2 \beta \text{Tr} \left( \Sigma_x + \Sigma^\dagger_x \right) \)

We can study the limit \( \beta = \frac{f^2}{4} \rightarrow \infty \) (\( f \) pion decay constant)

\[ \text{Tr} \left( \Sigma_x + \Sigma^\dagger_x \right) = 4 - \frac{2}{f^2} \pi_x^i \pi_x^i \quad \text{and} \quad \text{Tr} \left( \partial_\mu \Sigma_x \partial_\mu \Sigma^\dagger_x \right) = \frac{2}{f^2} \partial_\mu \pi_x^i \partial_\mu \pi_x^i \]

\[ S = \frac{1}{2} \int d^4 x \left\{ \pi_x^i (-\partial_\mu \partial_\mu + m_\pi^2) \pi_x^i + \phi_x (-\partial_\mu \partial_\mu + m_\phi^2) \phi_x + \lambda \phi_x \partial_\mu \pi_x^i \partial_\mu \pi_x^i + \lambda f^2 \nu \phi_x \right\} \]
OUR MODEL: on the lattice

\[ S = \sum_x \left\{ -\beta \text{Tr} \left[ \sum_{\mu} \left( \Sigma^\dagger x+\mu + \Sigma^\dagger x-\mu \right) \right] - \hat{m}_\pi^2 \beta \text{Tr} \left( \Sigma_x + \Sigma^\dagger_x \right) + \frac{1}{2} \left( 8 + \hat{m}_\phi^2 \right) \hat{\phi}_x^2 \right. \\
- \frac{1}{2} \hat{\phi}_x \sum_{\mu} \left( \hat{\phi}_x+\mu + \hat{\phi}_x-\mu \right) - \lambda \beta \hat{\phi}_x \text{Tr} \left[ \Sigma_x \sum_{\mu} \left( \Sigma^\dagger x+\mu + \Sigma^\dagger x-\mu \right) - \nu \right] \right\} \]

Both \( \phi_x \) and \( \Sigma_x \) can be updated using the heatbath algorithm

**Correlators**

\[ \Sigma = c_0 \mathbb{I} + c_i \sigma_i \quad \pi^i_x = \frac{f}{2i} \text{Tr} [\sigma_i \Sigma] = c_i^x f \quad f = 2 \sqrt{\beta} \quad (i = 1, 2, 3) \]

\[ \langle \hat{\pi}^i_x \hat{\pi}^i_y \rangle = 4 \hat{\beta} \langle c_i^x c_i^y \rangle \quad \langle \hat{\pi}^i_x \hat{\phi}_y \rangle = 2 \sqrt{\beta} \langle c_i^x \hat{\phi}_y \rangle \quad \langle \hat{\pi}^i_x \hat{\pi}^i_x \hat{\phi}_y \rangle = 4 \hat{\beta} \langle c_i^x c_i^x \hat{\phi}_y \rangle \]
\[ Z = \int D\Sigma D\phi \ e^{-S\Sigma}e^{-\sum_{x,y}[\frac{1}{2}\phi_x K_{xy} \phi_y] - \sum_{x} \phi_x b_x} \]

where \( K_{xy} = (-\partial_{\mu}\partial_{\mu} + m^2_{\phi})\delta_{xy} \) and \( b_x = \lambda \beta \text{Tr} \left( \partial_{\mu} \sum_x \partial_{\mu} \Sigma^\dagger_x + \nu \right) \)

Therefore:

\[
\langle \phi_x \phi_y \rangle = \frac{1}{Z} \frac{\partial^2 Z}{\partial b_x \partial b_y} = \langle (K^{-1})_{xy} \rangle + \langle (K^{-1} b)_x (K^{-1} b)_y \rangle
\]

\[
\langle \phi_x O_y \rangle = -\langle (K^{-1} b)_x O_y \rangle
\]

\[ K^{-1}_{nm} = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \tilde{G}(k) e^{ik(n-m)} \quad \tilde{G}(k) = \frac{1}{4 \sum_{\mu} \sin^2 \frac{k_{\mu}}{2} + m^2_{\phi}} \]

\[
C_x = \sum_y K^{-1}_{xy} b_y \Rightarrow \text{(Convolution theorem)} \Rightarrow \tilde{C}(k) = \tilde{G}(k)\tilde{b}(k) = \frac{\tilde{b}(k)}{4 \sum_{\mu} \sin^2 \frac{k_{\mu}}{2} + m^2_{\phi}}
\]
\[ \tilde{\pi}(\vec{n}, t) = \frac{1}{V} \sum_x \pi(\vec{x}, t)e^{i\vec{x}\vec{p}} \quad p_i = \frac{2\pi}{L_i} n_i \quad n_i = 0, \cdots, L_i - 1 \]

We introduce operators with zero total momentum and zero isospin:

\[ O_{\vec{n}}(t) = \sum_{i=1}^{3} \tilde{\pi}^i(\vec{n}, t)\tilde{\pi}^i(-\vec{n}, t) \quad \text{NOTE: } \tilde{\pi}^i(-\vec{n}, t) = [\tilde{\pi}^i(\vec{n}, t)]^* \]

In particular (in a cubic lattice):

\[ O_{n^2=0}(t) = \sum_{i=1}^{3} \tilde{\pi}^i(0, 0, 0, t)\tilde{\pi}^i(0, 0, 0, t) \]

\[ O_{n^2=1}(t) = \frac{1}{3} \left[ \sum_{i=1}^{3} \tilde{\pi}^i(1, 0, 0, t)\tilde{\pi}^i(-1, 0, 0, t) + \tilde{\pi}^i(0, 1, 0, t)\tilde{\pi}^i(0, -1, 0, t) + \tilde{\pi}^i(0, 0, 1, t)\tilde{\pi}^i(0, 0, -1, t) \right] \]

We consider \( n^2 = 0, 1, 2, 3, 4 \):

\( (0, 0, 0), (1, 0, 0), (1, 1, 0), (1, 1, 1), (2, 0, 0) \) (all non equivalent permutations and signs)

\[ O^*(t) = \tilde{\phi}(t) = \frac{1}{V} \sum_x \phi(\vec{x}, t)e^{i\vec{x}\vec{p}} \]
Correlation matrix function \((i, j = 1, \cdots, R)\): \(C_{ij}(t) = \langle O_i O_j \rangle\)

Generalized eigenvalues problem:

\[
C(t)\psi_\alpha = \lambda_\alpha(t, t_0)C(t_0)\psi_\alpha \quad \lambda_\alpha(t, t_0) = e^{-(t-t_0)m_\alpha}
\]

The pions are in a stationary scattering state; if they were free:

\[
E_n = 4 \sinh^{-1} \left[ \frac{1}{2} \sqrt{m_\pi^2 + 4 \sin^2 \left( \frac{\pi}{L_x} n_x \right) + 4 \sin^2 \left( \frac{\pi}{L_y} n_y \right) + 4 \sin^2 \left( \frac{\pi}{L_z} n_z \right)} \right]
\]
TUNING OF PARAMETERS

\[ E = 2 \sqrt{m_\pi^2 + 4 \sum_{i=1}^{3} \sin^2\left(\frac{\pi}{L_i} n_i\right)} \]

\[ 2m_\pi < m_\phi < 4m_\pi \quad m_\phi = km_\pi \quad \Rightarrow \quad m_\pi = \frac{4}{k^2 - 4} \sqrt{\sum_{i=1}^{3} \sin^2\left(\frac{\pi}{L_i} n_i\right)} \]

\[ m_\pi = 0.410 \]

\[ m_\phi = 9.0 \]

\[ \nu = 4.54 \]

\[ \text{data 120000} \]

\[ m_{\pi}^R = 0.468(1) \]

\[ \beta = 0.25 \]

\[ \lambda = 5.0 \]
If \( \beta \) is large \( \Rightarrow \) Vacuum Stability Problem (\( \beta > 0.25 \))

If \( \beta \) is small \( \Rightarrow \) Saturation effect in the gap energy

\[
C_{\phi\phi}(t) = \int_{-\pi}^{\pi} e^{i\Omega t} \frac{1}{4\sin^2 \frac{\Omega}{2} + m_\phi^2 - c\lambda^2 I(L, m_\pi, \Omega)}
\]

\[
4\sin^2 \frac{\Omega}{2} + m_\phi^2 - c\lambda^2 I(L, m_\pi, \Omega) = 0
\]

If we increase \( \lambda \) \( \Rightarrow \) we have to increase \( m_\phi \)
(because \( m_\phi^r \) must be fixed; \( (\Omega_1 + \Omega_2)/2 \) fixed)

If \( m_\phi, \lambda \gg 1 \):

\[
m_\phi^2 - c\lambda^2 I(L, m_\pi, \Omega) \approx 0 \quad \frac{m_\phi^2}{\lambda^2} \approx c I(L, m_\pi, \Omega^*) \quad \Rightarrow \quad \Omega_2 - \Omega_1 = \text{fixed}
\]

Should we think back to the model?
QCD CASE

\[ \langle G(t)|G(0) \rangle \quad \text{Glueball-Glueball prop} \]

\[ \langle G(t)|\pi(\vec{0}, 0)\pi(\vec{0}, 0) \rangle \quad \text{Glueball-Pions prop} \]

\[ \langle \pi(\vec{0}, t)\pi(\vec{0}, t)|\pi(\vec{0}, 0)\pi(\vec{0}, 0) \rangle \quad \text{Pions-Pions prop} \]

We can tune our model to obtain the correlators with the same (QCD) precision and therefore we can predict the precision of the (QCD) resonance parameters.
Summary, outlook

- The problem of resonances on the lattice
- Probability distribution method
- We are testing it on a model

Qualitative: We will use what we have learnt in the case of QCD

Quantitative: We can use our numerical results to predict the precision of what we will obtain in QCD