The $1/N_c$ Expansion in Baryons

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Abstract. The $1/N_c$ expansion in baryons is discussed with two applications, namely baryon masses and partial decay widths. These applications provide the basic insights on the utility of the expansion.

Keywords: Baryons, $1/N$ expansion

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INTRODUCTION

QCD at low and intermediate energies has a few possible expansion parameters. The parameters associated with quarks are the light quark masses and the inverse of the heavy quark masses. There is only one expansion parameter beyond these, namely $1/N_c$, where $N_c$ is the number of colors [1]. The former expansions have been put to good use in ChPT and in HQET. The $1/N_c$ expansion has been shown to work very well in the meson and glueball sectors, explaining the OZI rule, in the $1/N_c$ scalings in ChPT which are clearly shown in the LECs, in the scaling of the string tension and glueball masses, in the prediction of the quenched topological susceptibility via the $\eta'$ mass, etc. The extension of the $1/N_c$ expansion to the baryon sector was proposed back in 1979 [2], motivating first the revival of the Skyrme model [3], and later on, thanks to the discovery of the dynamical spin-flavor symmetry underlying the $1/N_c$ expansion [4], it became a well defined tool for analyzing ground state [5] as well as excited baryons [6, 7, 8]. A remarkable feature of the $1/N_c$ expansion is that it can be formulated at hadronic level thanks to the possibility of establishing well defined rules to set up the $1/N_c$ power counting for any observable, and thus making it applicable and useful for phenomenology.

The implementation of the expansion in baryons relies on the emergent dynamical spin-flavor $SU(2N_f)$ symmetry in large $N_c$ ($N_f$ is the number of light flavors), which is actually a contracted symmetry [4]. This symmetry is a result of consistency and it is broken by $1/N_c$ corrections. According to this, baryons are classified in multiplets of $SU(6)$, providing in this way a justification to the $SU(6)$ framework that has been used since the mid 1960’s, except that now it is known how to systematically implement the breaking of this symmetry consistently with QCD. These multiplets depend on $N_c$, being $56$-, $70$- or $20$-plets at $N_c = 3$. Other multiplets, such as exotic baryons, could also be implemented, although there is no experimental evidence of such states.

The $1/N_c$ expansion is set up through the representation of amplitudes or observables of interest in a series of effective operators ordered in powers of $1/N_c$. For instance, a given QCD operator can be represented at the baryon level by a series of the form:

$$O_{QCD} \rightarrow \sum_n C_n N_c^{1-n} O_n$$

where $O_n$ is an effective $n$-body operator acting at the baryon level, and the coefficients $C_n$ containing the QCD dynamics are determined by fitting to experimental data or to lattice results. The matrix elements of the operator $O_n$ have also a $1/N_c$ power counting determined entirely by the spin-flavor structure of the operator. In principle, as one carries out the expansion, the coefficients $C_n$, which start at $O(N_c^0; m_q^0)$ will have corrections in powers of $1/N_c$ and the singlet component of the quark masses $(m_u + m_d + m_s)$. This representation in effective operators can be extended to amplitudes, such as strong transitions, helicity amplitudes, etc.
In general, even at subleading order in $1/N_c$, the number of effective operators will be (much) smaller than the number of possible data, and thus predictions and tests are possible. For instance, if one carries out the expansion to first subleading order, the accuracy will $O(1/N_c^2)$ or about 10% in practice. The $1/N_c$ expansion also provides parameter free relations valid at a given order in $1/N_c$; such relations must be satisfied by QCD as well as any model that respects the $1/N_c$ power counting, such as the Skyrme model and certain versions of the quark model. Testing the validity of these relations represents a direct test of how well the $1/N_c$ is faring.

Note that the coefficients $C_n$ contain the whole dynamics associated with the corresponding effective operator. Since this includes short as well as long distance dynamics, it is at this stage difficult to understand the underlying physics associated with each coefficient. In some cases the quark model picture seems to give a reasonable first description, e.g., for hyperfine splitting operators. In other cases it seems likely that both long and short distance physics give important contributions, representing a challenge for any simple physical picture. As we mention below, lattice QCD will provide important help, in particular allowing for a determination of quark mass dependencies of coefficients.

In this paper we present applications to baryon masses, strong decays. We comment on other possible applications.

**MASSES**

**Ground state baryon masses**

The GS baryons fill the totally symmetric $SU(6)$ representation, which is the $56$-plet at $N_c = 3$ and contains the $J=1/2$ octet and the $J=3/2$ decuplet. In the case of baryon masses one builds the effective operators in terms of the generators of the spin-flavor group. The most general mass formula valid to $O(1/N_c)$ and linear order in quark masses is [5]:

$$M_{GS} = C_1 N_c + \frac{C_{HF}}{N_c} \left(S^2 - \frac{3}{4} N_c\right) - C_8 m_s - m_{u,d} \frac{S}{\Lambda}$$

(2)

where $S$ is the spin operator, and $S$ is the strangeness. Other operators contributing at $O(1/N_c)$ are reducible [5]. The leading $O(N_c)$ operator gives the spin-flavor singlet mass. Note that Eqn. (2) is in fact the old Gürsey-Radicati mass formula! The higher power corrections to the mass formula are spelled out in Ref. [9].

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**FIGURE 1.** The lower lying states have mass splittings $O(1/N_c)$, while the higher state splittings are $O(N_c^0)$.

The coefficients $C_1$, $C_{HF}$, and $C_8$ are obtained by a best fit to the octet and decuplet masses. The higher order analysis [9] has further shown the expected rate of convergence.

The parameter free mass relations shown in Table 1 result from the mass formula. These relations receive corrections $O(1/N_c^2; m_s^2/N_c)$ and chiral loop corrections $O(m_s^4/\Lambda^4)$. The relations pertaining the $1/N_c$ expansion are the last two, and they are well satisfied.

<table>
<thead>
<tr>
<th>Table 1. Mass relations for GB baryons. GMO: Gell-Mann Okubo, ES: equal spacing. Last column shows the tests of the relations.</th>
</tr>
</thead>
<tbody>
<tr>
<td>GMO</td>
</tr>
<tr>
<td>ES</td>
</tr>
<tr>
<td>$\Omega^+ - \Xi_{10} = \Xi_{10} - \Xi_{10}$</td>
</tr>
<tr>
<td>8-10</td>
</tr>
<tr>
<td>$3\Lambda + \Sigma_8 - 2(N + \Sigma_8) = \Xi_{10} + \Sigma_{10} - \Omega^+ - \Delta$</td>
</tr>
</tbody>
</table>

**Excited baryon masses**

In order to describe excited baryons, in addition to spin-flavor one needs the spatial degrees of freedom that can be excited and which are conveniently classified in terms of the representations of the rotation group $O(3)$. Although the
$O(3)$ is not a symmetry in large $N_c$ limit, it turns out to be a better than expected symmetry in practice, and thus it is convenient to use a basis of states belonging to irreducible representations of $SU(6) \times O(3)$, a practice used since the 1960’s. In the strict large $N_c$ limit excited baryons fill irreducible representations of a contracted spin-flavor symmetry [8, 10]. This happens even when the $SU(6) \times O(3)$ is broken at $O(N_c^0)$.

For illustration we give a brief overview of the masses of the lightest negative parity baryons. They fit into a $[70, l^p = 1^-]$, and consist of the following states [11]: two $8_j = 1/2$, two $8_{3/2}$, and one of each $8_{5/2}$, $10_{1/2}$, $10_{3/2}$, $11_{1/2}$, and $1_{3/2}$. This is the most completely known excited multiplet, with 18 out of 30 isospin multiplets experimentally established. The masses of the $70$-plet were studied in [11, 12], to which we refer for details. One approximation made, expected to be good, is that there is no configuration mixing, i.e., mixing of states in different $SU(6) \times O(3)$.

A discussion of the issues surrounding this approximation is given in Ref. [13]. The basis of mass operators consists of the $SU(6)$ singlet $O(N_c)$ operator, four $O(N_c^0)$ operators (responsible for splitting states into towers of grand-spin [8, 10]), seven $SU(3)$ invariant $O(1/N_c)$ operators and finally three $SU(3)$ breaking operators $O(m_i - m_{u,d})$. Fitting to the known masses, one finds [11] a number of important insights: i) the $O(N_c)$ operators give small contributions as compared to the dominant hyperfine operator of $O(1/N_c)$, giving thus a small breaking of the $O(3)$. ii) the magnitude of sub-leading corrections is always natural, i.e. within the magnitude expected by the $1/N_c$ power. iii) the dominant operators are the leading order one and the hyperfine operator, others tend to have coefficients that are smaller or much smaller than the natural size; this is indication of the underlying dynamics being to a first approximation similar to the one of quark models. iv) the spin-orbit puzzle in the quark models, represented by the nearly vanishing mass splitting between all spin-orbit partners except the $SU(3)$ singlet baryons $\Lambda(1405)$ and $\Lambda(1520)$, can be naturally resolved. The splitting between the latter states is only due to the spin-orbit operator, while the splittings in the other spin-orbit partners receive in addition contributions from the other two $O(N_c)$ operators involving couplings to $\ell$, giving sufficient freedom for resolving the puzzle.

One important test is provided by the two mixing angles $\theta_{1,2}$ associated with the two pairs of octets. These mixing angles can be obtained from the masses and independently from partial decay widths [14, 15] and helicity amplitudes [16], and all three determinations are consistent. We note that the mixing angles at large $N_c$ have definite values, namely $\theta_1 = 54.7^\circ$ and $\theta_2 = 65.9^\circ$, to be compared with the ones obtained in the analyses, $\sim 25 \pm 10^\circ$ and $\sim 161 \pm 10^\circ$. This discrepancy is simply because the $O(N_c^0)$ mass operators turn out to give small contributions to the masses. In a sense the discrepancy is a result of the smallness of the $O(3)$ symmetry breaking.

In addition to GMO and ES relations, one finds six parameter free mass relations involving combinations of masses across the $SU(3)$ multiplets in the $70$-plet [11]. These relations give predictions for the masses of the missing states in the $70$-plet. For instance, only one $\Xi(1820)$ state is known in the $70$-plet, and the masses of six remaining states are predicted by those relations. Searching for these states, for instance in double K-meson photo-production at Jefferson Lab, would provide an important test of the whole framework.

An interesting work [17] has studied the quark model along with the $1/N_c$ expansion. In the $70$-plet this study leads to two relations, valid when 3-body operators are neglected, which are independent of the quark model potential. These relations involve the two mixing angles $\theta_{1,3}$, and turn out to be well satisfied by the values mentioned above [12, 11, 6, 14, 15].

The masses of other multiplets have been analyzed as well [18], and the general conclusion is that throughout no clear indication of a violation of the hierarchy expected from $1/N_c$ power counting has been found. Note that higher multiplets with masses above 2 GeV have a small fraction of the states experimentally established, and therefore the conclusions are not very strong in those cases.

Comments

Two issues should be commented on. The first is the mentioned configuration mixing. This type of mixing is in general suppressed by power of $1/\sqrt{N_c}$ except in cases where it can be $O(N_c^0)$ [13]. In the latter cases it was shown that the operators producing the mixing require couplings to orbital degrees of freedom, which are expected to be small if one assumes that the suppression observed in the spin-orbit couplings within multiplets extends to multiplet mixings [13]. If this suppression takes place, perturbation theory shows that the effect can be simply absorbed into the coefficients of the mass expansion. The second and more difficult issue is that excited baryon widths, unlike the case of mesons, are $O(N_c^0)$. Since the widths can be rather large in many cases, there remain some problems about the definition of the baryon masses. One needs to adopt a definition of the masses (we have used Breit-Wigner masses and the corresponding values from PDG). In principle, one can be more rigorous as a theoretically consistent $1/N_c$ analysis at the level of the T-matrix [27] can be implemented. This in particular indicates that the constraints imposed by the $1/N_c$ power counting can in principle be included in partial wave analyses.
Regge trajectories

Understanding the nearly linear baryon Regge trajectories can be further improved by using the mass formulas. The idea is to isolate the piece of the mass formulas which provide the linear behavior. One expects that this piece will be the leading order spin-flavor singlet term in the mass formulas. Taking known baryons up to masses of $\sim 2.5$ GeV, and assigning states into $56$- and $70$-plets, a fit to the masses to $O(1/N_c; m_s)$ provides the spin-flavor singlet component of the masses $M_0(\ell) = N_c C_1(\ell)$. Plotting $M_0^2(\ell)$ vs $\ell$ one finds two separate and linear Regge trajectories [20] with slightly different slopes. The separation between trajectories is an effect $O(N_0^0)$ in $M_0$, and can be visualized as due to the difference between the exchange interactions in the $56$- and $70$-plets. This result has several important implications: i) it emphasizes the importance of the approximate $O(3)$ symmetry giving $\ell$ a key role; ii) indicates small spin-flavor configuration mixing effects. Note that the usually depicted trajectories using the actual baryon masses split the shown trajectories by hyperfine effects mostly. Hyperfine effects give deviations from linear behavior in general, and thus those "physical" trajectories are not so clear, in particular not allowing the discrimination of $56$- from $70$-plet trajectories.

FIGURE 2. $M_0^2 = (N_c C_1)^2$ vs $\ell$ for the $56$-plets (red) and the $70$-plets (blue).

Masses from the lattice

Recently the masses of ground state baryons have been calculated on the lattice by various groups, such as Refs. [21, 22], for several different quark masses. Rapid progress is taking place in calculations of the excited baryon spectrum as well [23]. These results represent an excellent opportunity to further test the $1/N_c$ expansion. The test of parameter free mass relations on the GS baryon masses from the lattice has recently carried out [24]. Another interesting aspect is the determination of the quark mass dependence of the effective coefficients. As a preliminary result, we show here the coefficients $C_1$, $C_{HF}$ in Eqn. (2) as fitted to various lattice results. The figure shows the trend of coefficients as a function of the pion mass squared; the expected approximately linear growth of $C_1$ and a slight decrease of hyperfine coefficient $C_{HF}$. The assumption that the HF interaction is to a large extent given by one gluon exchange means that it is of short range nature mostly, and thus it should be expected that it will not be affected very significantly by long distance effects as one approaches the physical limit. This seems to be the case for both lattice inputs we considered here.

FIGURE 3. Physical quark masses in black, and lattice results from [21] (red) and [22] (blue).
An analysis for the lattice excited baryon spectrum can be carried out and is currently under consideration [25].

DECAYS

The partial decay widths are, along with the masses, one of the quantities carrying key information about excited baryon properties. Here we briefly describe the analysis of the partial widths, describing the case of positive parity baryon decays. The decays of the negative parity 70-plet are presented in [15].

![Diagram showing strong transitions from excited to GS multiplet via pseudoscalar emission.](image)

**FIGURE 4.** Strong transitions from excited to GS multiplet via pseudoscalar emission.

The decay amplitudes for emission of a single pseudoscalar meson are given by

\[
M(\ell_\pi, Y_\pi, I_\pi) = (-1)^{I_\pi} \sqrt{2M_B} \frac{\sqrt{N_c}}{F_\pi} \langle B_{GS} | B^{\ell_\pi Y_\pi I_\pi} | B^* \rangle
\]

where \( \pi \) represents a member of the pseudoscalar octet with flavor quantum number \( (Y_\pi, I_\pi) \), \( \ell_\pi \) is its angular momentum, \( B^* \) and \( B \) are the excited and GS baryon respectively, and \( B^{\ell_\pi Y_\pi I_\pi} \) is the transition operator to be expanded in effective operators:

\[
B^{\ell_\pi Y_\pi I_\pi} = \left( \frac{k_\pi}{\Lambda} \right)^{\ell_\pi} \sum_n C_n O_n^{\ell_\pi Y_\pi I_\pi}, \quad O_n^{\ell_\pi Y_\pi I_\pi} = [\xi^\ell G_n^{j_\pi j_\ell I_\pi}]^\dagger
\]

\( G_n \) are spin-flavor tensor operators with \( j_n \) the corresponding operator spin, \( \xi^\ell \) is the \( \ell \)-rank O(3) tensor with \( \ell \) the \( O(3) \) quantum number of excited baryon, and \( C_n \) are the coefficients to be fitted. A centrifugal barrier term has been included with \( \Lambda \) arbitrary scale.

We describe as illustration the decays of the positive parity \([56, 0^+]\) (Roper multiplet) and \([56, 2^+]\), referring for details to [26]. The analysis described here includes effects of \( SU(3) \) breaking in the amplitudes. The operator basis is shown in Table 2.

**TABLE 2.** \( SU(6) \) effective operators for 56-plet decays.

<table>
<thead>
<tr>
<th>Operator ( G_1 )</th>
<th>Order ( O(N_c^0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_2 ) ( \equiv \frac{1}{N_c} {S, G}[j=1] )</td>
<td>( O(1/N_c) )</td>
</tr>
<tr>
<td>( G_3 ) ( \equiv \frac{1}{N_c} {S, G}[j=2] )</td>
<td>( O(1/N_c) )</td>
</tr>
<tr>
<td>( G_{SB} ) ( \equiv \frac{1}{N_c} \left( d_{0ab} - \delta_{ab}/\sqrt{3} \right) G_{ib} )</td>
<td>( O((m_s - m_{u,d})/\sqrt{N_c}) )</td>
</tr>
</tbody>
</table>

### \([56, 0^+]\) decays

Here we show the results for the decays of the Roper multiplet. The partial widths are described to \( O(1/N_c) \) and \( O(m_s - m_{u,d}) \) by two \( SU(3) \) preserving and one \( SU(3) \) violating operators. Normalizing the operators to have natural size matrix elements, the coefficients of the fit to the seven partial widths as given by the PDG are shown in Table 3. The coefficients are of natural size, however we observe that the LO partial widths of the \( N(1440) \) and \( \Delta(1600) \) are poorly described. The NLO correction, which is given by the 2-body operator, is essential for describing those widths. Table 3 shows the results and comparison with empirical widths. The following relations are valid at LO:

\[
\begin{align*}
\Gamma(N(1440) \to \pi N)/\Gamma(N(1440) \to \pi \Delta) &= 7.5 \text{ vs } 2.6 \pm 2.1 \text{ Exp} \\
\Gamma(\Delta(1600) \to \pi N)/\Gamma(\Delta(1600) \to \pi \Delta) &= 1.7 \text{ vs } 0.3 \pm 0.2 \text{ Exp} \\
\Gamma(N(1440) \to \pi N)/\Gamma(\Delta(1600) \to \pi \Delta) &= 1.64 \text{ vs } 1.10 \pm 0.63 \text{ Exp} \text{ also valid at NLO!}
\end{align*}
\]
The first two relations show the need for the NLO corrections. The third, which is valid also at NLO, is satisfied within errors. This is indeed an encouraging fact about the $1/N_c$ analysis.

One general result in the decays of $[56,0^+]$-plets is that the non-strange baryons have partial widths $O(1/N_c^2)$ into an $\eta$ in the final state. In fact, such a suppression seems to be well satisfied in the known decays of the $[56,0^+]$-plets analyzed here. The presence of a large or significant $\eta$ decay mode is therefore an indicator that the non-strange excited baryon has an important $[70]$-plet component, since the $\eta$ channel is not suppressed in $[70]$-plets. Clearly, the observed suppression is consistent with small configuration mixing effects.

$[56,2^+]$ P-wave decays

Here we show the results for the P-wave partial widths of the $[56,2^+]$. The F-wave decays and other details can be found in [26]. For the P-waves, one has seven empirical inputs, and the fits are shown in Table 4. One finds excellent fit using only the LO operator and the $SU(3)$ breaking operator. Notice however that the seven input empirical widths have large errors, which limits the possibility of obtaining significant information about the NLO effects, giving very large relative errors in the coefficients $C_{2,3}$.

<table>
<thead>
<tr>
<th>TABLE 3.</th>
<th>Operator coefficients for the P-wave $[56,0^+]$ decays and partial decay widths of known decay channels. $C_i$ are the coefficients of $SU(3)$ preserving operators and $B_1$ the coefficient of the $SU(3)$ breaking operator.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$N_{1/2}(1440)$</td>
</tr>
<tr>
<td></td>
<td>$\pi N$</td>
</tr>
<tr>
<td>LO</td>
<td>148</td>
</tr>
<tr>
<td>NLO</td>
<td>214</td>
</tr>
<tr>
<td>Error</td>
<td>48.8</td>
</tr>
<tr>
<td>Exp</td>
<td>211(88)</td>
</tr>
</tbody>
</table>

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<tr>
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</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$N_{2/3}(1720)$</td>
</tr>
<tr>
<td></td>
<td>$\eta N$</td>
</tr>
<tr>
<td>LO</td>
<td>36</td>
</tr>
<tr>
<td>NLO</td>
<td>27</td>
</tr>
<tr>
<td>Error</td>
<td>6.1</td>
</tr>
<tr>
<td>Exp</td>
<td>34(16)</td>
</tr>
</tbody>
</table>

In general, NLO corrections can be determined at a natural level of accuracy only if the inputs have errors $O(1/N_c^2)$, i.e., $\sim 10$% (double as much for decay widths). One immediately notices that the empirical errors in decay widths are
significantly larger than that. In particular, this means that the predictions for the unknown decay widths are not very accurate. Establishing the importance of $SU(3)$ breaking effects in the decay amplitudes requires additional empirical input to the available one. In the case of the 70-plet decays the situation is similar, but there is stronger evidence for the need of those symmetry breaking effects [15] than in the 56-plets.

OTHER APPLICATIONS

Other very interesting applications have been carried out. Among them are the applications to the EM properties. The magnetic moments and form factors have been studied for the GS baryons [28, 29]. The photo-production helicity amplitudes for positive and negative parity baryons have also been studied [16], and these analyses provide very interesting insights into the consistency of the expansion when both strong and EM transitions are considered, as well as on some particular helicity amplitudes which quark models systematically fail to describe. In the GS sector the $1/N_c$ expansion has been combined with baryon ChPT [30], where the inclusion of the $\Delta$ resonance in chiral loops becomes a consistent procedure. There are exciting applications still to be done using this marriage of baryon ChPT and the $1/N_c$ expansion.

COMMENTS

The $1/N_c$ expansion is a natural tool in QCD. For the skeptic $N_c = 3$ is too small, and for the optimist it is large enough. We cannot prove who is definitely right for now, but we can carry out the procedure of analyzing known phenomenology and find out whether the expansion seems to work or not. We think that there are reasons to feel as an optimist based on the applications that have been made in both mesons and baryons.

The nice aspect about the $1/N_c$ expansion is that it can be formally implemented at hadronic level in the spirit of effective theories. The price to be payed is that the unknown QCD dynamics sitting in the coefficients of the expansion must be extracted from phenomenology or from lattice simulations, and it is still a long way until we can sort out that dynamics. Nonetheless, an important level of insight can be obtained.

Perhaps one of the most exciting opportunities for applying the expansion is to lattice results. Currently these applications can be done to baryon masses, as decay widths on the lattice are still a problem for the future. The lattice brings the possibility of quantifying quark mass dependencies of the expansion coefficients, and thus may be helpful for understanding some aspects of the dynamics they encode.

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