Properties of the proton and neutron in the quark model

A good way to introduce the ideas encoded in the quark model is to understand how it simply explains properties of the ground-state baryons and mesons

The most common of those, although not necessarily the simplest, are the proton and neutron

To understand how to explain their magnetic moments in the quark model, we have to understand their spin and flavor structure, the exchange group S_3, isospin, color, and how these can be used to evaluate matrix elements of operators

Properties of the Proton

Proton has mass $1.67262158 \times 10^{-27}$ kg

In more useful units: 938 MeV/c^2, or just 938 MeV (use c = 1)

Its charge is $+1.602 \times 10^{-19}$ C, or +e

It is a fermion, with J = 1/2

It has a magnetic moment of $+2.79\mu_N = +2.79\dfrac{e\hbar}{2m_p}$

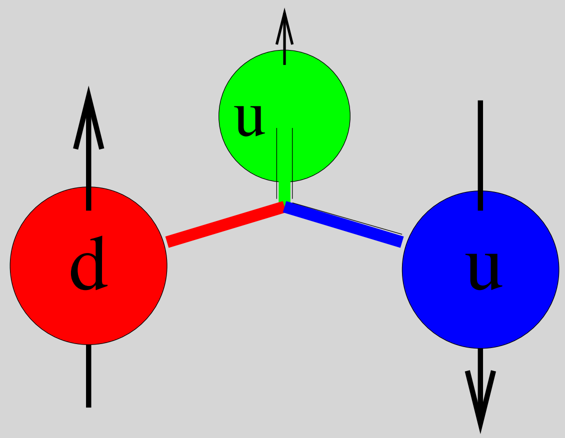
It is not point-like; charge radius is 0.88×10^{-15} m = 0.88 fm

If you invert the coordinates (**r** to −**r**) of its constituents, wave function stays the same; it has positive intrinsic (coordinate inversion) parity

$$\Psi(-\mathbf{r}_1, -\mathbf{r}_2, \ldots) = +\Psi(\mathbf{r}_1, \mathbf{r}_2, \ldots)$$

Fundamental constituents of the Proton

Most of the proton's properties are determined by its three valence quarks



up (u) quark is a fundamental spinor (fermion with J=1/2), charge +2/3 e

down (d) quark is also a fermion with J=1/2, charge −1/3 e

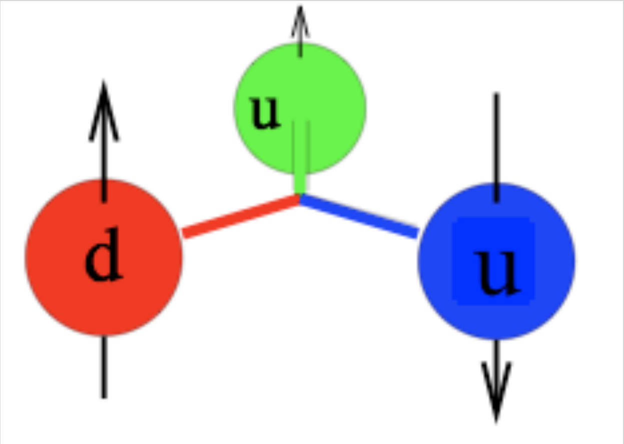
Because of the strength of the interaction binding the quarks to each other, the proton can contain any number of quark-antiquark pairs, called (Dirac) sea quarks

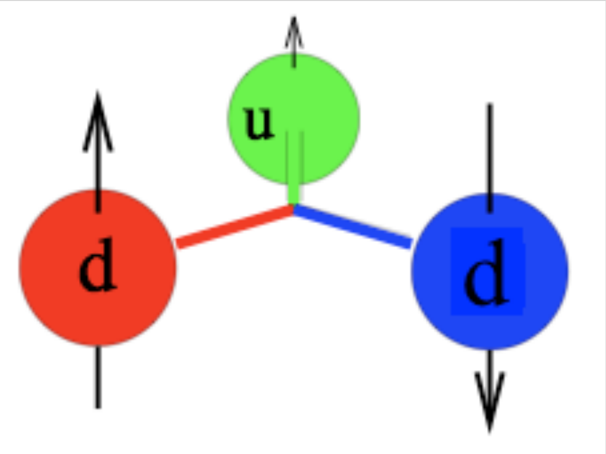
The exchange quantum of this interaction is called a gluon, and has vector quantum numbers; the proton can contain any number of gluons

Excitations that change the quarks

These are called different ‘flavors’

Simplest: change one of the up quarks for a down quark





Neutron has mass ~1.3 MeV larger than the proton:

d quark has mass ~ 5 MeV larger than the u

electrostatic repulsion weaker in the neutron:

$2(+2/3)(-1/3)+(-1/3)(-1/3) = -1/3$ (units $e^2/\langle r\rangle$)

than in the proton:

$2(+2/3)(-1/3)+(+2/3)(+2/3) = 0$, lowers mass by ~3 MeV

Properties of the Neutron

Neutron decays slowly (886 s) via beta (weak) decay to proton: $n \rightarrow p \; e^- \;\; \bar{\nu}_e$

down quark has decayed to an up: $d \rightarrow u \; W^- \rightarrow u \; e^- \;\; \bar{\nu}_e$

Has $J^P = 1/2^+$ like the proton

Magnetic moment is $-1.91 \; \mu_N = -1.91 \; \dfrac{e\hbar}{2m_p}$

Neutral, but has a *distribution* of charge, with

$$\langle \sum_i e_i r_i^2 \rangle = -0.12 \text{ fm}^2$$

negative charge outside a positive core; up quark on average inside the two down quarks

Nucleon (p and n) magnetic moments

$SU(3)_f$ symmetry showed that baryons were made up of three 'valence' quarks which carried their quantum numbers

Natural to assume these quarks are spin-½ objects, like all other fundamental matter particles

What are the possible spin wave functions of three spin-½ objects?

$$
\begin{aligned}
(1/2 \otimes 1/2) \otimes 1/2 &= (0 \oplus 1) \otimes 1/2 \\
&= 1/2 \oplus (1 \otimes 1/2) \\
&= (1/2)_\rho \oplus (1/2)_\lambda \oplus 3/2
\end{aligned}
$$

We can determine the spin wave functions using the rules for combining representations of SU(2) encoded in the Clebsch-Gordan coefficients (e.g., from the Particle Data Group table):

Spin wave functions

$$
\begin{aligned}
(1/2 \otimes 1/2) \otimes 1/2 &= (0 \oplus 1) \otimes 1/2 \\
&= 1/2 \oplus (1 \otimes 1/2) \\
&= (1/2)_\rho \oplus (1/2)_\lambda \oplus 3/2
\end{aligned}
$$

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

	J	J	...
	M	M	...
m_1 m_2	Coefficients		
m_1 m_2			
⋮ ⋮			

$1/2 \times 1/2$

m_1 m_2	$J=1$, $M=+1$	$J=1$, $M=0$	$J=0$, $M=0$	$J=1$, $M=-1$
+1/2 +1/2	1			
+1/2 −1/2		1/2	1/2	
−1/2 +1/2		1/2	−1/2	
−1/2 −1/2				1

$1 \times 1/2$

m_1 m_2	$J=3/2$, $M=+3/2$	$J=3/2$, $M=+1/2$	$J=1/2$, $M=+1/2$	$J=3/2$, $M=-1/2$	$J=1/2$, $M=-1/2$	$J=3/2$, $M=-3/2$
+1 +1/2	1					
+1 −1/2		1/3	2/3			
0 +1/2		2/3	−1/3			
0 −1/2				2/3	1/3	
−1 +1/2				1/3	−2/3	
−1 −1/2						1

2×1

	$J=3$, $M=+3$	$J=3$	$J=2$

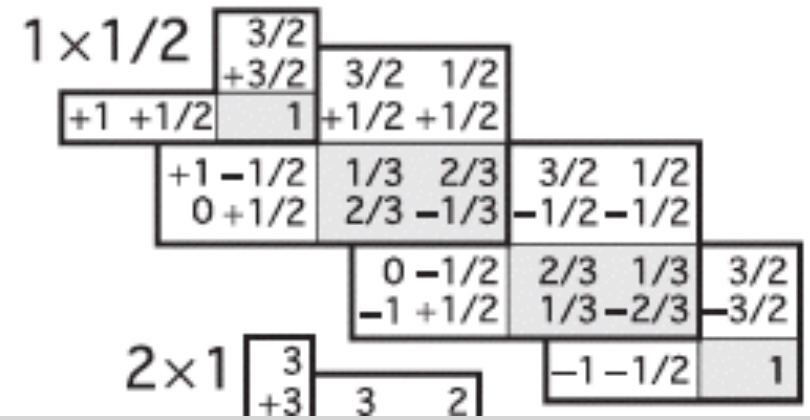
Define $\uparrow := |\tfrac12\ \tfrac12\rangle$ $\downarrow := |\tfrac12\ -\tfrac12\rangle$, then

$$|0\ 0\rangle = (1/\sqrt{2})(\uparrow\downarrow - \downarrow\uparrow)$$

$$|1\ 1\rangle = \uparrow\uparrow,\ |1\ 0\rangle = (1/\sqrt{2})(\uparrow\downarrow + \downarrow\uparrow),\ |1\ -1\rangle = \downarrow\downarrow$$

Spin wave functions

$$(1/2 \otimes 1/2) \otimes 1/2 = (0 \;\oplus\; 1) \otimes 1/2$$
$$= 1/2 \oplus (1 \otimes 1/2\,)$$
$$= (1/2)_\rho \oplus (1/2)_\lambda \oplus 3/2$$



$$|3/2\ 3/2\rangle = \uparrow\uparrow\uparrow$$
$$|3/2\ 1/2\rangle = (1/\sqrt{3})|1\ 1\rangle\downarrow + \sqrt{(2/3)}|1\ 0\rangle\uparrow$$
$$= (1/\sqrt{3})(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow)$$
$$|3/2\ {-1/2}\rangle = \sqrt{(2/3)}|1\ 0\rangle\downarrow + (1/\sqrt{3})|1\ {-1}\rangle\uparrow$$
$$= (1/\sqrt{3})(\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow + \downarrow\downarrow\uparrow)$$
$$|3/2\ 3/2\rangle = \downarrow\downarrow\downarrow$$

These wave functions are *totally symmetric* under exchange of the quarks

Exchange symmetry

This is no accident: they are eigenfunctions of total spin

$$\mathbf{S}^2 = (\mathbf{S}_1+\mathbf{S}_2+\mathbf{S}_3)^2 = s_1(s_1+1) + s_1(s_1+1) + s_1(s_1+1)$$
$$+ 2(\mathbf{S}_1\cdot\mathbf{S}_2 + \mathbf{S}_1\cdot\mathbf{S}_3 + \mathbf{S}_2\cdot\mathbf{S}_3)$$
$$= 9/4 + 2(\mathbf{S}_1\cdot\mathbf{S}_2 + \mathbf{S}_1\cdot\mathbf{S}_3 + \mathbf{S}_2\cdot\mathbf{S}_3)$$

and $\mathbf{S}_z = (\mathbf{S}_1+\mathbf{S}_2+\mathbf{S}_3)_z$

The group of all exchange operations $\mathbf{O}_{S3}$ for three objects is called S_3: consists of the identity, the two-cycles (12), (13), (23), and the three-cycles (123) and (132)

$\mathbf{S}^2$ and $\mathbf{S}_z$ are totally symmetric under exchange, so $[\mathbf{S}^2, \mathbf{O}_{S3}] = [\mathbf{S}_z, \mathbf{O}_{S3}] = 0$ and all simultaneous eigenfunctions of $\mathbf{S}^2$ and $\mathbf{S}_z$ must *represent* the operators of S_3 (i.e., must belong to a *representation* of the group)

Exercise : exchange symmetry

Prove, by considering their effect on *abc*, that the three-cycles (123) and (132) can be written as the product of two two-cycles

Exchange symmetry

$|S\, M_S\rangle$ satisfies $\mathbf{S}^2\, |S\, M_S\rangle = S(S+1)\, |S\, M_S\rangle$, i.e. is an eigenfunction of the operator $\mathbf{S}^2$

Act on both sides with any operator $\mathbf{O}_{S3}$ of the exchange group S_3

$\mathbf{O}_{S3}\, \mathbf{S}^2\, |S\, M_S\rangle = S(S+1)\, \mathbf{O}_{S3}|S\, M_S\rangle$
$\mathbf{S}^2\, (\mathbf{O}_{S3}|S\, M_S\rangle) = S(S+1)\, \mathbf{O}_{S3}|S\, M_S\rangle$ since commutator is zero

$\mathbf{O}_{S3}|S\, M_S\rangle$ is *also* an eigenfunction of $\mathbf{S}^2$ with the *same* eigenvalue (similarly for $\mathbf{S}_z$)

S=3/2 case: there is only one eigenfunction of $\mathbf{S}^2$ and $\mathbf{S}_z$ with the eigenvalues $S(S+1)$ and M_S, then $\mathbf{O}_{S3}|S\, M_S\rangle$ is *the same* as $|S\, M_S\rangle$, up to a phase, +1 in the case of $|3/2\, M_S\rangle$

Exchange symmetry

$$\begin{aligned}(1/2 \otimes 1/2) \otimes 1/2 &= (0 \oplus 1) \otimes 1/2 \\ &= 1/2 \oplus (1 \otimes 1/2) \\ &= (1/2)_\rho \oplus (1/2)_\lambda \oplus 3/2\end{aligned}$$

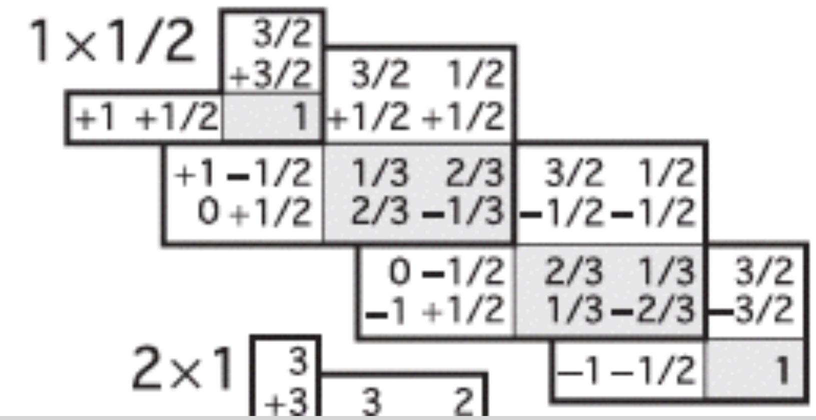
$$\mathbf{S}^2 (\mathbf{O}_{S3}|S\ M_S\rangle) = S(S+1)\ \mathbf{O}_{S3}|S\ M_S\rangle$$

There are two S = ½ eigenfunctions $|1/2\ M_S\rangle$ of $\mathbf{S}^2$, so $\mathbf{O}_{S3}|S\ M_S\rangle$ is in general a linear combination of these two eigenfunctions that conserves probability (i.e., a rotation)

The two eigenfunctions $|1/2\ M_S\rangle$, which we will denote $|1/2_\rho\ M_S\rangle$ and $|1/2_\lambda\ M_S\rangle$, form a two-dimensional *mixed-symmetry* representation of S_3

Spin wave functions

$$(1/2 \otimes 1/2) \otimes 1/2 = (0 \oplus 1) \otimes 1/2$$
$$= 1/2 \oplus (1 \otimes 1/2)$$
$$= (1/2)_\rho \oplus (1/2)_\lambda \oplus 3/2$$

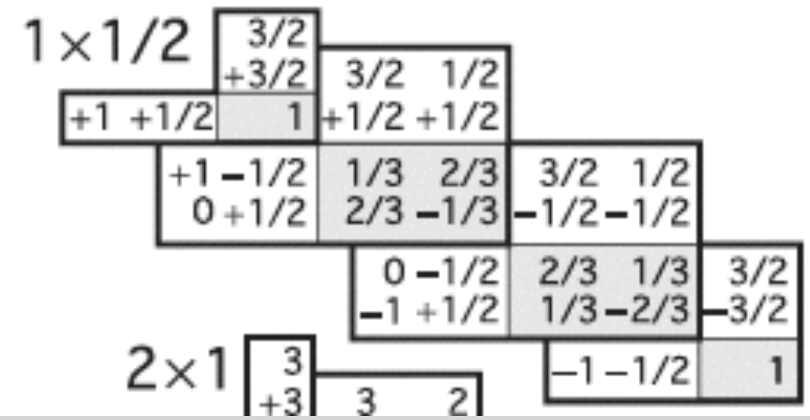


$$|1/2\ 1/2\rangle_\rho = |0\ 0\rangle \uparrow$$
$$= (1/\sqrt{2})(\uparrow\downarrow - \downarrow\uparrow) \uparrow$$
$$= (1/\sqrt{2})(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

$$|1/2\ {-1/2}\rangle_\rho = |0\ 0\rangle \downarrow$$
$$= (1/\sqrt{2})(\uparrow\downarrow - \downarrow\uparrow) \downarrow$$
$$= (1/\sqrt{2})(\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow)$$

Spin wave functions

$$(1/2 \otimes 1/2) \otimes 1/2 = (0 \ \oplus \ 1) \otimes 1/2$$
$$= 1/2 \oplus (1 \otimes 1/2)$$
$$= (1/2)_\rho \oplus (1/2)_\lambda \oplus 3/2$$



$$|1/2\ 1/2\rangle_\lambda = \sqrt{(2/3)}|1\ 1\rangle \downarrow - (1/\sqrt{3})|1\ 0\rangle \uparrow$$
$$= \sqrt{(2/3)}\uparrow\uparrow\downarrow - (1/\sqrt{6})\uparrow\downarrow\uparrow - (1/\sqrt{6})\downarrow\uparrow\uparrow$$
$$= -(1/\sqrt{6})(\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow - 2\uparrow\uparrow\downarrow)$$

$$|1/2\ {-1/2}\rangle_\lambda = (1/\sqrt{3})|1\ 0\rangle \downarrow - \sqrt{(2/3)}|1\ {-1}\rangle \uparrow$$
$$= (1/\sqrt{6})\uparrow\downarrow\downarrow - (1/\sqrt{6})\downarrow\uparrow\downarrow - \sqrt{(2/3)}\downarrow\downarrow\uparrow$$
$$= (1/\sqrt{6})(\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow - 2\downarrow\downarrow\uparrow)$$

Exchange symmetry

The two eigenfunctions $|1/2\ M_S\rangle$, denoted $|1/2_\rho\ M_S\rangle$ and $|1/2_\lambda\ M_S\rangle$, form a two-dimensional *mixed-symmetry* representation of $S_3 = \{1,(12),(13),(23),(123),(132)\}$

$$1\ |1/2\ 1/2\rangle_\rho = |1/2\ 1/2\rangle_\rho$$

$$(12)\ |1/2\ 1/2\rangle_\rho = (12)\ (1/\sqrt{2})(\uparrow\downarrow - \downarrow\uparrow)\uparrow = -\ |1/2\ 1/2\rangle_\rho$$

$$\begin{aligned}(13)\ |1/2\ 1/2\rangle_\rho &= (13)\ (1/\sqrt{2})(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)\\ &= (1/\sqrt{2})(\uparrow\downarrow\uparrow - \uparrow\uparrow\downarrow)\\ &= (1/\sqrt{2})[2\uparrow\downarrow\uparrow + (-\downarrow\uparrow\uparrow + \downarrow\uparrow\uparrow) - 2\uparrow\uparrow\downarrow]/2\\ &= (1/\sqrt{2})(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow - 2\uparrow\uparrow\downarrow)/2\\ &= (1/2)\ |1/2\ 1/2\rangle_\rho - \sqrt{6}/(2\sqrt{2})\ |1/2\ 1/2\rangle_\lambda\\ &= (1/2)\ |1/2\ 1/2\rangle_\rho - (\sqrt{3}/2)\ |1/2\ 1/2\rangle_\lambda\end{aligned}$$

(a rotation by −60°)

Exercise: exchange symmetry

a. Prove that $(13)\,|1/2\ 1/2\rangle_\lambda$ is the orthogonal linear combination $-(\sqrt{3}/2)\,|1/2\ 1/2\rangle_\rho - (1/2)\,|1/2\ 1/2\rangle_\lambda$

b. Without doing an explicit calculation, can you find $(23)\,|1/2\ 1/2\rangle_\rho$ using the action of (12) on $|1/2\ 1/2\rangle_\rho$ and on $|1/2\ 1/2\rangle_\lambda$ and knowing that
$(13)\,|1/2\ 1/2\rangle_\rho = (1/2)\,|1/2\ 1/2\rangle_\rho - (\sqrt{3}/2)\,|1/2\ 1/2\rangle_\lambda$?

Nucleon (p and n) magnetic moments

In order to calculate the magnetic moments of the spin-½ nucleons N={n,p} we need to know how each quark's charge is correlated with its spin (assume M_S = +½)

$$\boldsymbol{\mu}_N = \langle N\ \tfrac{1}{2}\ \tfrac{1}{2}|\ \textstyle\sum_i \boldsymbol{\mu}_{qi}\ |N\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle$$
$$= \langle N\ \tfrac{1}{2}\ \tfrac{1}{2}|\ \textstyle\sum_i q_i\, \mathbf{S}_{zi}/(2\, m_i)\ |N\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

This is equivalent to knowing the flavor (correlated with charge, since $q_u = +\,2/3$ and $q_d = -\,1/3$) and spin parts of the wave function

The strong interactions are almost identical for up and down quarks due to their very similar masses $m_d - m_u \simeq 5$ MeV compared to the scale of the strong interactions $\Lambda_{QCD} \simeq$ 200 MeV or the proton mass 938 MeV

Isospin symmetry

This gives rise to a symmetry of the strong interactions

Broken only at the ~1% scale, by quark-mass differences and electromagnetic interactions between the quarks

Even though quarks can only be up or down, the strong interactions are invariant under a *rotation*

$$\begin{aligned} u &= \cos(\theta)\, u + \sin(\theta)\, d \\ d &= -\sin(\theta)\, u + \cos(\theta)\, d \end{aligned}$$

This approximate SU(2) symmetry is called *isospin*

Isospin wave functions of $u = |\, \tfrac{1}{2}\ \tfrac{1}{2}\rangle$ and $d = |\, \tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle$ are the exact equivalent of the $\uparrow \coloneqq |\, \tfrac{1}{2}\ \tfrac{1}{2}\rangle$ and $\downarrow \coloneqq |\, \tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle$ spin-½ wave functions

Isospin symmetry

We already know the isospin (flavor) wave functions of the proton and neutron!

They are an almost degenerate doublet so

$|I\ I_3\rangle = |1/2\ 1/2\rangle$ for p, $|1/2\ {-1/2}\rangle$ for n

Must be isospin-½ combination of three isospin-½ objects, of which there are two mixed-symmetry variants:

$$\varphi^p_\rho = (1/\sqrt{2})(udu - duu)$$
$$\varphi^p_\lambda = (1/\sqrt{6})(udu + duu - 2\ uud)$$

$$\varphi^n_\rho = (1/\sqrt{2})(udd - dud)$$
$$\varphi^n_\lambda = (1/\sqrt{6})(udd + dud - 2\ ddu)$$

Isospin symmetry

If isospin symmetry is real, there should also be a quartet of states with

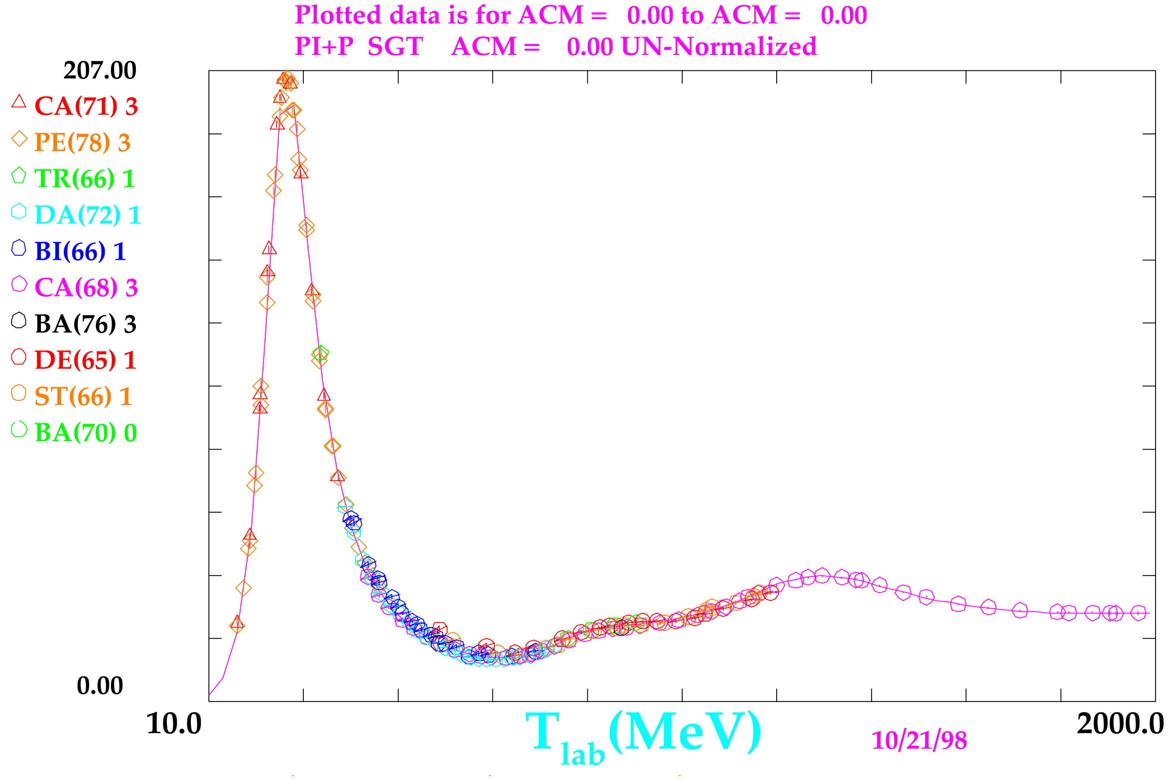
$$|I\ I_3\rangle = |3/2\ 3/2\rangle, |3/2\ 1/2\rangle, |3/2\ -1/2\rangle, |3/2\ -3/2\rangle$$

$$\varphi^{\Delta^{++}} = uuu$$
$$\varphi^{\Delta^{+}} = (1/\sqrt{3})(uud + udu + duu)$$
$$\varphi^{\Delta^{0}} = (1/\sqrt{3})(ddu + dud + udd)$$
$$\varphi^{\Delta^{-}} = ddd$$

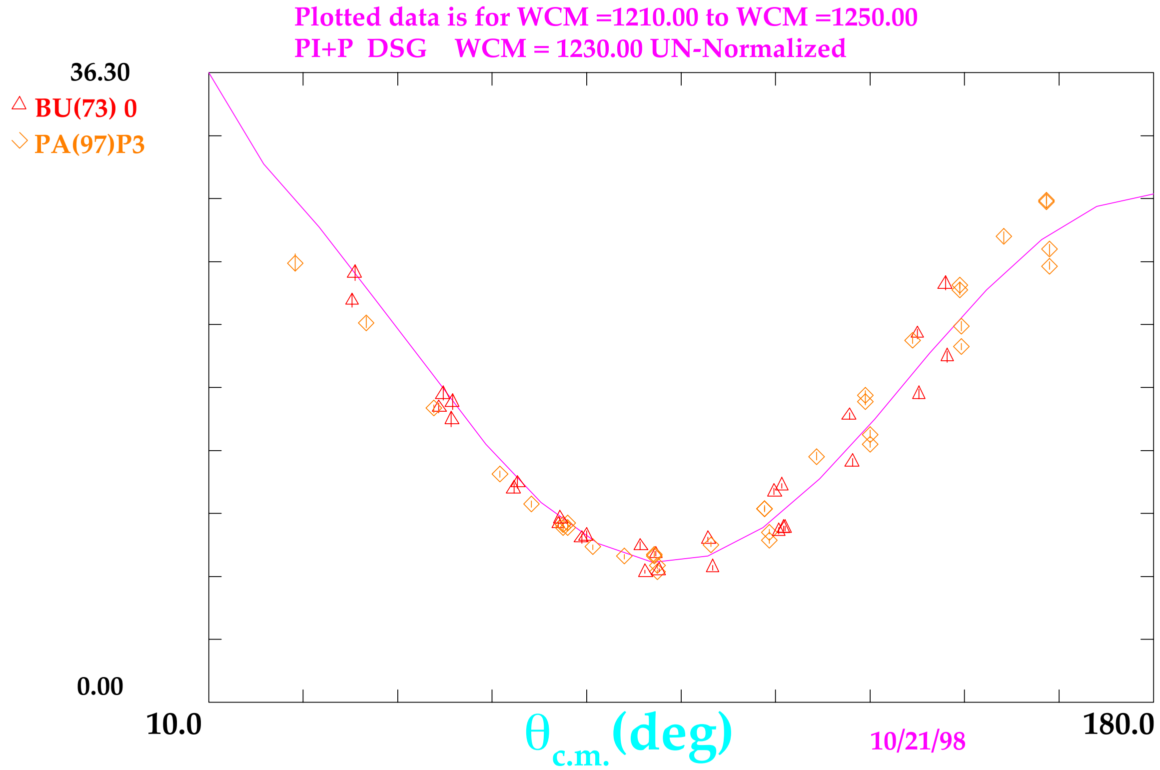
This is realized in nature in the form of strongly-decaying (and so extremely short-lived) particle called the Δ, which has mass ~1232 MeV and decays $\Delta \to N\,\pi$, which can be thought of as an excited nucleon

Isospin symmetry

Study of the angular distribution of the decay products of the $\Delta^{++} \rightarrow p\,\pi^+$ shows a P-wave angular distribution



total $p\pi^+$ cross section



differential cross section

(Overall) spin and parity J^P for proton is $1/2^+$, for the pion is 0^-, so J^P for Δ is $(1/2^+)(0^-)$ in a P ($L^\pi=1^-$) wave, so it either has $J^P=1/2^+$ or $3/2^+$; it has $J^P = 3/2^+$

Exercise: isospin symmetry

a. The Δ^0 can be formed by the reaction $\pi^- p \rightarrow \Delta^0$, and it subsequently decays via $\Delta^0 \rightarrow \pi^- p$ or $\pi^0 n$ essentially 100% of the time. Since the decay is a strong decay, the decay operator must be approximately isospin-symmetric. What proportion of the final states are $\pi^- p$, and what proportion are $\pi^0 n$?

b. No other strong decays of Δ^0 are possible because $M_\Delta - M_N \sim 294$ MeV is much smaller than the mass of every meson except the pion. Can you think of another way the Δ^0 could decay? Can you estimate what fraction of the time it decays this way?

Exchange symmetry

There are three kinds of representations of S_3

One-dimensional totally symmetric (S) representations, e.g., the spin-3/2 or isospin-3/2 wave functions

Two-dimensional mixed-symmetry $\{M^\rho, M^\lambda\}$ representations that transform into each other, e.g., the pair $|1/2\ 1/2\rangle_\rho$ and $|1/2\ 1/2\rangle_\lambda$

One-dimensional totally anti-symmetric (A) representations

Exchange symmetry, the Pauli principle, and color

The Δ is the lightest of many excited states of the nucleon which have isospin 3/2, and so is likely in its spatial ground state; in other bound systems $L^P = 0^+$ ground states are exchange symmetric

If it has spin-3/2, its spin wave function is exchange symmetric

Consider Δ^{++}, with the isospin (flavor) wave function $\varphi^{\Delta^{++}} = uuu$, must be exchange symmetric

If these are the only degrees of freedom of the quarks, then there is a problem with the Pauli principle

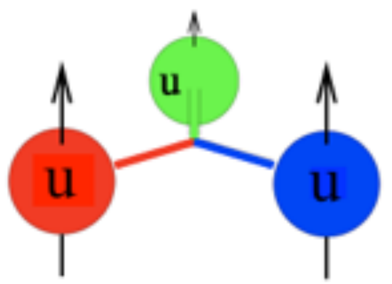
$$\Psi^{\Delta^{++}} = \psi^S(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)\ |S=3/2\ M_S\rangle\ |I=3/2\ I_3\rangle$$

Exchange symmetry, the Pauli principle, and color

A simple way out: propose that quarks carry a three-valued degree of freedom, called color; make this part of the wave function totally antisymmetric under quark exchange

$$C_A = (1/\sqrt{6})(rgb + brg + gbr - rbg - bgr - grb)$$

Under all actions of S_3 this transforms into itself up to a sign $\mathbf{O}_{S3}\, C_A = -\, C_A$ and so forms a 1-dimensional (antisymmetric) representation



Δ^{++} with $|S\ M_S\rangle = |3/2\ 3/2\rangle$

$\Psi^{\Delta^{++}} = C_A\, \psi^S(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)\ |S{=}3/2\ M_S\rangle\ |I{=}3/2\ I_3\rangle$ and the Pauli principle is intact!

Nucleon (p and n) wave functions

Focus for now on the proton; for convenience use notation $\chi^\rho = |1/2\ M_S\rangle_\rho$, $\chi^\lambda = |1/2\ M_S\rangle_\rho$

$$\Psi^p(M_S) = C_A\ \psi^S(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)\ \{\chi^\rho \text{ or } \chi^\lambda\}\ \{\phi^p{}_\rho \text{ or } \phi^p{}_\lambda\}$$

We need to arrange for the product of spin and isospin wave functions to be totally symmetric to satisfy the Pauli principle

There are the equivalent of the Clebsch-Gordan coefficients which tell us how to combine representations of S_3

Combining representations of S3 exchange group

Combining two representations:

$$S_a \otimes S_b = S_{ab}$$
$$S_a \otimes A_b = A_{ab}$$
$$A_a \otimes A_b = S_{ab}$$

$$\frac{1}{\sqrt{2}}(M_a^\rho \otimes M_b^\rho + M_a^\lambda \otimes M_b^\lambda) = S_{ab}$$
$$\frac{1}{\sqrt{2}}(M_a^\rho \otimes M_b^\lambda - M_a^\lambda \otimes M_b^\rho) = A_{ab}$$
$$\frac{1}{\sqrt{2}}(M_a^\rho \otimes M_b^\lambda + M_a^\lambda \otimes M_b^\rho) = M_{ab}^\rho$$
$$\frac{1}{\sqrt{2}}(M_a^\rho \otimes M_b^\rho - M_a^\lambda \otimes M_b^\lambda) = M_{ab}^\lambda$$

Nucleon (p and n) wave functions

The symmetric combination of spin and isospin wave functions is

$$(1/\sqrt{2})[\chi^{\rho}(M_S)\ \varphi^{p}{}_{\rho} + \chi^{\lambda}(M_S)\ \varphi^{p}{}_{\lambda}]$$

The totally anti-symmetric proton wave function is

$$\Psi^{p}(M_S) = C_A\ \psi^{S}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)(1/\sqrt{2})[\chi^{\rho}(M_S)\ \varphi^{p}{}_{\rho} + \chi^{\lambda}(M_S)\ \varphi^{p}{}_{\lambda}]$$

and similarly for the neutron with $\varphi^{p} \rightarrow \varphi^{n}$

Exercise: proton wave function

a. Prove, by writing out expanding $[\chi^\rho(M_S)\, \varphi^p{}_\rho + \chi^\lambda(M_S)\, \varphi^p{}_\lambda]$ into 13 terms like $u_\uparrow d_\downarrow u_\uparrow$, that it is symmetric under any exchange (12), (13), or (23)

b. Similarly, prove that $[\chi^\rho(M_S)\, \varphi^p{}_\lambda - \chi^\lambda(M_S)\, \varphi^p{}_\rho]$ is totally antisymmetric

Nucleon (p and n) magnetic moments

We now have all the ingredients we need to calculate

$$\boldsymbol{\mu}_N = \langle N\ ½\ ½|\ \sum_i q_i\ \mathbf{S}_{zi}/(2\ m_i)\ |N\ ½\ ½\rangle$$

Let's start with the proton; we can get the neutron result by the transformation $u \leftrightarrow d$

$$\begin{aligned}\boldsymbol{\mu}_p = {} & C_A^\dagger C_A \int d^3\mathbf{r}\ |\psi^S(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)|^2\ (1/2)[\chi^\rho(½)\ \varphi^p{}_\rho + \chi^\lambda(½)\ \varphi^p{}_\lambda]^\dagger \\ & \qquad \sum_i q_i\ \mathbf{S}_{zi}/(2\ m_i)\ [\chi^\rho(½)\ \varphi^p{}_\rho + \chi^\lambda(½)\ \varphi^p{}_\lambda] \\ = {} & (1/2)[\chi^\rho(½)\ \varphi^p{}_\rho + \chi^\lambda(½)\ \varphi^p{}_\lambda]^\dagger \\ & \qquad \sum_i q_i\ \mathbf{S}_{zi}/(2\ m_i)\ [\chi^\rho(½)\ \varphi^p{}_\rho + \chi^\lambda(½)\ \varphi^p{}_\lambda]\end{aligned}$$

Nucleon (p and n) magnetic moments

The combined spin-flavor wave function $(1/\sqrt{2})[\chi^\rho(\tfrac{1}{2})\, \varphi^p{}_\rho + \chi^\lambda(\tfrac{1}{2})\, \varphi^p{}_\lambda]$ is totally symmetric, so the result is three times the magnetic moment of quark 3

$$\boldsymbol{\mu}_p = (3/2)\, [\chi^\rho(\tfrac{1}{2})\, \varphi^p{}_\rho + \chi^\lambda(\tfrac{1}{2})\, \varphi^p{}_\lambda]^\dagger \; \boldsymbol{\mu}_{q3}\, [\chi^\rho(\tfrac{1}{2})\, \varphi^p{}_\rho + \chi^\lambda(\tfrac{1}{2})\, \varphi^p{}_\lambda]$$

The operator q_3 only measures the third quark's charge and can't change the flavor of *any* quark

$$\varphi^p{}_\rho{}^\dagger\, \boldsymbol{\mu}_{q3}\, \varphi^p{}_\rho = (1/2)(udu-duu)^\dagger\, \boldsymbol{\mu}_{q3}\, (udu-duu) = \boldsymbol{\mu}_u$$

$$\varphi^p{}_\lambda{}^\dagger\, \boldsymbol{\mu}_{q3}\, \varphi^p{}_\lambda = (1/6)(udu+duu-2uud)^\dagger\, \boldsymbol{\mu}_{q3}\, (udu+duu-2uud) = (1/6)(\boldsymbol{\mu}_u + \boldsymbol{\mu}_u + 4\boldsymbol{\mu}_d)$$

Nucleon (p and n) magnetic moments

$$\boldsymbol{\mu}_p = 3\,[\chi^\rho(\tfrac{1}{2})\,\varphi^p{}_\rho + \chi^\lambda(\tfrac{1}{2})\,\varphi^p{}_\lambda]^\dagger$$
$$\qquad\qquad \boldsymbol{\mu}_{q3}\,[\chi^\rho(\tfrac{1}{2})\,\varphi^p{}_\rho + \chi^\lambda(\tfrac{1}{2})\,\varphi^p{}_\lambda]$$

$$\varphi^p{}_\rho{}^\dagger\,\boldsymbol{\mu}_{q3}\,\varphi^p{}_\rho = \boldsymbol{\mu}_u,\;\; \varphi^p{}_\lambda{}^\dagger\,q_3\,\varphi^p{}_\lambda = (1/6)(\boldsymbol{\mu}_u + \boldsymbol{\mu}_u + 4\boldsymbol{\mu}_d)$$

$\boldsymbol{\mu}_{q3}$ can't change the (12) symmetry of the wave function:

$$\varphi^p{}_\lambda{}^\dagger\,\boldsymbol{\mu}_{q3}\,\varphi^p{}_\rho = (1/\sqrt{2})(udu-duu)^\dagger\,\boldsymbol{\mu}_{q3}\,(1/\sqrt{6})(udu+duu-2uud)$$
$$= 0$$
$$\varphi^p{}_\rho{}^\dagger\,\boldsymbol{\mu}_{q3}\,\varphi^p{}_\lambda = 0$$

$$\boldsymbol{\mu}_p = 3\,\chi^\rho(\tfrac{1}{2})^\dagger\,\boldsymbol{\mu}_{3u}\,\chi^\rho(\tfrac{1}{2})$$
$$\qquad + 3\,\chi^\lambda(\tfrac{1}{2})^\dagger\,(1/6)(\boldsymbol{\mu}_{3u} + \boldsymbol{\mu}_{3u} + 4\boldsymbol{\mu}_{3d})\,\chi^\lambda(\tfrac{1}{2})$$

Nucleon (p and n) magnetic moments

$$\boldsymbol{\mu}_p = 3\,[\chi^\rho(\tfrac{1}{2})^\dagger\, \boldsymbol{\mu}_{3u}\, \chi^\rho(\tfrac{1}{2}) + \chi^\lambda(\tfrac{1}{2})^\dagger\, (1/6)(\boldsymbol{\mu}_{3u} + \boldsymbol{\mu}_{3u} + 4\boldsymbol{\mu}_{3d})\, \chi^\lambda(\tfrac{1}{2})]$$

$$\chi^\rho(\tfrac{1}{2})^\dagger\, \mathbf{S}_{3z}\, \chi^\rho(\tfrac{1}{2}) = (1/2)(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)^\dagger\, \mathbf{S}_{3z}\, (\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) = +1/2$$

$$\chi^\lambda(\tfrac{1}{2})^\dagger\, \mathbf{S}_{3z}\, \chi^\lambda(\tfrac{1}{2}) = (1/6)(\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow - 2\uparrow\uparrow\downarrow)^\dagger\, \mathbf{S}_{3z}\, (\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow - 2\uparrow\uparrow\downarrow) = (1/6)[+1/2 + 1/2 + 4(-1/2)] = -1/6$$

Putting this all together, we have

$$\boldsymbol{\mu}_p = (3/2)\boldsymbol{\mu}_u + 3(1/6)(-1/6)(\boldsymbol{\mu}_u + \boldsymbol{\mu}_u + 4\boldsymbol{\mu}_d) = (4/3)\boldsymbol{\mu}_u - (1/3)\boldsymbol{\mu}_d$$

Which implies $\boldsymbol{\mu}_n = (4/3)\boldsymbol{\mu}_d - (1/3)\boldsymbol{\mu}_u$

Nucleon (p and n) magnetic moments

$$\boldsymbol{\mu}_p = (4/3)\boldsymbol{\mu}_u - (1/3)\boldsymbol{\mu}_d, \quad \boldsymbol{\mu}_n = (4/3)\boldsymbol{\mu}_d - (1/3)\boldsymbol{\mu}_u$$

Solve for $\boldsymbol{\mu}_u$: $4\boldsymbol{\mu}_p + \boldsymbol{\mu}_n = (16/3 - 1/3)\ \boldsymbol{\mu}_u$

$$\boldsymbol{\mu}_u = (4\boldsymbol{\mu}_p + \boldsymbol{\mu}_n)/5$$

Similarly (or by isospin) $\boldsymbol{\mu}_d = (4\boldsymbol{\mu}_n + \boldsymbol{\mu}_p)/5$

Experimentally, $\mu_p = 2.79\ \mu_N = 2.79\ e/(2\ m_N)$

$\mu_n = -1.91\ \mu_N$

If the quark model has any validity, then we should have

$$\mu_u = \mu_N[4(2.79)-1.91]/5 = +1.85\ \mu_N$$

$$\mu_d = \mu_N[4(-1.91)+2.79]/5 = -0.97\ \mu_N$$

Nucleon (p and n) magnetic moments

$$\mu_u = \mu_N[4(2.79)-1.91]/5 = +1.85\ \mu_N$$

$$\mu_d = \mu_N[4(-1.91)+2.79]/5 = -0.97\ \mu_N$$

These are very close to the ratio +2/3 : −1/3 of the up and down quark charges

If we assume $\mu_u = (+2/3)e/(2m_{u,d})$, $\mu_d = (-1/3)e/(2m_{u,d})$ then the light-quark mass needs to be approximately $m_{u,d} \simeq m_N/3$

This observation was an early success of the quark model; corrected by relativity and the presence of quark-antiquark pairs in the nucleon

Nucleon (p and n) magnetic moments

Questions?