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COMPLETE EXPERIMENTS IN PSEUDOSCALAR PHOTOPRODUCTION

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Necessary and sufficient conditions are derived for three double polarisation measurements to complement cross section and single polarisation measurements in pseudoscalar meson photoproduction to enable amplitudes to be determined up to discrete ambiguities. Rules for choosing two further measurements to resolve the discrete ambiguities are given and practical applications of the rules to particular reactions are discussed.

1. Introduction

There have been many investigations, both theoretical and experimental, of pseudoscalar meson photoproduction off nucleons. In recent years there has been a large increase in polarisation information, by using both polarised beams and polarised targets and by measuring recoil polarisations. For example in the reaction $\gamma p \to \pi^0 p$ we now have data on differential cross sections, $d\sigma/dt$, linearly polarised photon asymmetries, Σ, recoil baryon polarisations, P, and polarised target asymmetries, T *. Proposals have been made to carry out measurements of double polarisation parameters by the simultaneous use of a polarised beam *and* a polarised target [3]. It is well known that we need 7 measurements to determine the amplitudes up to an overall phase and up to discrete ambiguities. It is clearly of interest to be able to decide whether a given set of three double polarisation measurements will enable an amplitude analysis to be carried out when taken in conjunction with $d\sigma/dt$, T, P and Σ. There is confusion in the existing literature on this point. In a recent publication Goldstein et al. [4] give a set of rules for deciding whether a set of seven measurements give complete information **. Their rules are in contradiction with earlier work by Worden [5] in that it appears from ref. [4] that one can obtain a complete set of measurements with a polarised target and polarised beams. Worden [5] claims that this is not so, but does not give any prescription for obtaining a complete set. Other work [6] on the subject is incomplete. To clarify this situation

* See ref. [1] for a list of high-energy data and ref. [2] for low-energy data.
** As always, up to quadrant ambiguities.

we have derived *necessary and sufficient* conditions for three double polarisation measurements to complement the information given by $d\sigma/dt$, P, Σ and T. (These four measurements we call the set S.) The conditions can be stated as follows.

There are twelve double polarisation measurements which give new information. In addition double polarisation measurements using a target polarised perpendicular to the reaction plane or measuring recoil polarisation perpendicular to the reaction plane are equivalent to single polarisation measurements. The twelve new measurements can be divided into three sets of four, the sets being characterised as beam-target (BT), target-recoil (TR) and beam-recoil (BR) with obvious connotations. Then a necessary and sufficient condition that three measurements give complete information up to an overall phase and up to discrete ambiguities when taken together with $d\sigma/dt$, Σ, P and T is that the three measurements are not all taken from the same set.

To eliminate the discrete ambiguities we show that two further measurements will suffice, provided that of the five double polarisation measurements performed, no four come from the same set

In sect. 2 we give the formalism necessary for deriving our results, in sect. 3 we give a proof of our main result, in sect. 4 we discuss discrete ambiguities and in sect. 5 we discuss implications.

2. Formalism

Although the result can be stated economically without any formalism to derive it we must define amplitudes. It turns out that the derivation in terms of transversity amplitudes is both transparent and instructive. We first define s-channel helicity amplitudes N, S_1, S_2 and D as in a previous paper [1] where N is a no-flip amplitude S_1 and S_2 single-flip amplitudes and D a double flip amplitude. We then define the following transversity amplitudes *:

$$b_1 = \tfrac{1}{2}[(S_1 + S_2) + i(N - D)], \qquad b_3 = \tfrac{1}{2}[(S_1 - S_2) - i(N + D)],$$

$$b_2 = \tfrac{1}{2}[(S_1 + S_2) - i(N - D)], \qquad b_4 = \tfrac{1}{2}[(S_1 - S_2) + i(N + D)]. \tag{1}$$

To define our axes we adopt the usual Basel convention with the z-axis being the beam direction and the y axis the normal to the reaction plane (fig. 1). The z' axis is in the direction of the scattered meson. We now define the sixteen observables in terms of both helicity and transversity amplitudes in table 1. The precise relation between observables and the experiments we consider is as follows.

* b_1, b_2, b_3 and b_4 are Von Gehlen's [7] Y_+, Y_-, X_+ and X_- respectively apart from a factor.

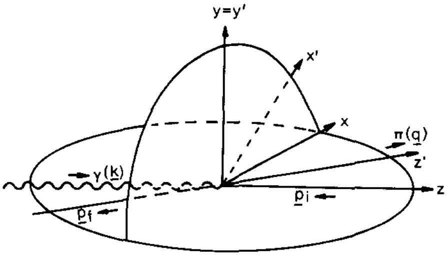

Fig. 1. Definition of axes. If $\boldsymbol{k}$ is the incoming photon momentum and $\boldsymbol{q}$ the outgoing meson momentum (both in the c.m. system) then the axes are defined by

$$z = \boldsymbol{k}/|\boldsymbol{k}|, \qquad y = \boldsymbol{k} \times \boldsymbol{q}/|\boldsymbol{k} \times \boldsymbol{q}|, \qquad x = y \times z,$$
$$z' = \boldsymbol{q}/|\boldsymbol{q}|, \qquad y' = y, \qquad x' = y \times z'.$$

Table 1
Observables

Usual symbol	Helicity representation	Transversity representation	Experiment required [a]	Type
$\mathrm{d}\sigma/\mathrm{d}t$	$\|N\|^2+\|S_1\|^2+\|S_2\|^2+\|D\|^2$	$\|b_1\|^2+\|b_2\|^2+\|b_3\|^2+\|b_4\|^2$	$\{-;-;-\}$	S
$\Sigma\,\mathrm{d}\sigma/\mathrm{d}t$	$2\mathrm{Re}(S_1^* S_2 - ND^*)$	$\|b_1\|^2+\|b_2\|^2-\|b_3\|^2-\|b_4\|^2$	$\{L(\frac{1}{2}\pi,0);-;-\}$ $\{-;y;y\}$	
$\mathrm{T}\,\mathrm{d}\sigma/\mathrm{d}t$	$2\mathrm{Im}(S_1N^* - S_2D^*)$	$\|b_1\|^2-\|b_2\|^2-\|b_3\|^2+\|b_4\|^2$	$\{-;y;-\}$ $\{L(\frac{1}{2}\pi,0);0;y\}$	
$\mathrm{P}\,\mathrm{d}\sigma/\mathrm{d}t$	$2\mathrm{Im}(S_2N^* - S_1D^*)$	$\|b_1\|^2-\|b_2\|^2+\|b_3\|^2-\|b_4\|^2$	$\{-;-;y\}$ $\{L(\frac{1}{2}\pi,0);y;-\}$	
$\mathrm{G}\,\mathrm{d}\sigma/\mathrm{d}t$	$-2\mathrm{Im}(S_1S_2^* + ND^*)$	$2\mathrm{Im}(b_1b_3^* + b_2b_4^*)$	$\{L(\pm\frac{1}{4}\pi);z;-\}$	BT
$\mathrm{H}\,\mathrm{d}\sigma/\mathrm{d}t$	$-2\mathrm{Im}(S_1D^* + S_2N^*)$	$-2\mathrm{Re}(b_1b_3^* - b_2b_4^*)$	$\{L(\pm\frac{1}{4}\pi);x;-\}$	
$\mathrm{E}\,\mathrm{d}\sigma/\mathrm{d}t$	$\|S_2\|^2-\|S_1\|^2-\|D\|^2+\|N\|^2$	$-2\mathrm{Re}(b_1b_3^* + b_2b_4^*)$	$\{c;z;-\}$	
$\mathrm{F}\,\mathrm{d}\sigma/\mathrm{d}t$	$2\mathrm{Re}(S_2D^* + S_1N^*)$	$2\mathrm{Im}(b_1b_3^* - b_2b_4^*)$	$\{c;x;-\}$	
$\mathrm{O}_x\,\mathrm{d}\sigma/\mathrm{d}t$	$-2\mathrm{Im}(S_2D^* + S_1N^*)$	$-2\mathrm{Re}(b_1b_4^* - b_2b_3^*)$	$\{L(\pm\frac{1}{4}\pi);-;x'\}$	BR
$\mathrm{O}_z\,\mathrm{d}\sigma/\mathrm{d}t$	$-2\mathrm{Im}(S_2S_1^* + ND^*)$	$-2\mathrm{Im}(b_1b_4^* + b_2b_3^*)$	$\{L(\pm\frac{1}{4}\pi);-;z'\}$	
$\mathrm{C}_x\,\mathrm{d}\sigma/\mathrm{d}t$	$-2\mathrm{Re}(S_2N^* + S_1D^*)$	$2\mathrm{Im}(b_1b_4^* - b_2b_3^*)$	$\{c;-;x'\}$	
$\mathrm{C}_z\,\mathrm{d}\sigma/\mathrm{d}t$	$\|S_2\|^2-\|S_1\|^2-\|N\|^2+\|D\|^2$	$-2\mathrm{Re}(b_1b_4^* + b_2b_3^*)$	$\{c;-;z'\}$	
$\mathrm{T}_x\,\mathrm{d}\sigma/\mathrm{d}t$	$2\mathrm{Re}(S_1S_2^* + ND^*)$	$2\mathrm{Re}(b_1b_2^* - b_3b_4^*)$	$\{-;x;x'\}$	TR
$\mathrm{T}_z\,\mathrm{d}\sigma/\mathrm{d}t$	$2\mathrm{Re}(S_1N^* - S_2D^*)$	$2\mathrm{Im}(b_1b_2^* - b_3b_4^*)$	$\{-;x;z'\}$	
$\mathrm{L}_x\,\mathrm{d}\sigma/\mathrm{d}t$	$2\mathrm{Re}(S_2N^* - S_1D^*)$	$2\mathrm{Im}(b_1b_2^* + b_3b_4^*)$	$\{-;z;x'\}$	
$\mathrm{L}_z\,\mathrm{d}\sigma/\mathrm{d}t$	$\|S_1\|^2+\|S_2\|^2-\|N\|^2-\|D\|^2$	$2\mathrm{Re}(b_1b_2^* + b_3b_4^*)$	$\{-;z;z'\}$	

[a] Notation is $\{P_\gamma; P_\mathrm{T}; P_\mathrm{R}\}$ where:
P_γ = polarisation of beam, $L(\theta)$ = beam linearly polarised at angle θ to scattering plane,
C = circularly polarised beam;
P_T = direction of target polarisation;
P_R = component of recoil polarisation measured.
In the case of the single polarisation measurements we also give the equivalent double polarisation measurement.

Polarised beam – polarised target [5]

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\mathrm{d}\sigma}{\mathrm{d}t}\Bigg|_{\text{unpolarised}} \{1 - P_\mathrm{T}\Sigma \cos(2\phi)$$

$$+ P_x\,[-P_\mathrm{T}H \sin(2\phi) + P_\odot F] - P_y\,[-T + P_\mathrm{T}P\cos(2\phi)] \tag{2}$$

$$- P_z\,[-P_\mathrm{T}G \sin(2\phi) + P_\odot E]\},$$

where (P_x, P_y, P_z) is the polarisation of the target, P_T is the transverse polarisation of the beam at an angle ϕ to the reaction plane and $P_\odot$ is the degree of right circular polarisation of the beam.

Beam-recoil [8]

$$\rho_\mathrm{f}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\mathrm{d}\sigma}{\mathrm{d}t}\Bigg|_{\text{unpolarised}} \{1 + \sigma_y P - P_\mathrm{T}\cos(2\phi)\,(\Sigma + \sigma_y T)$$

$$- P_\mathrm{T}\sin(2\phi)\,(O_x\sigma_x + O_z\sigma_z) - P_\odot(C_x\sigma_x + C_z\sigma_z)\}. \tag{3}$$

Target-recoil [8]

$$\rho_\mathrm{f}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\mathrm{d}\sigma}{\mathrm{d}t}\Bigg|_{\text{unpolarised}} \{1 + \sigma_y P + P_x(T_x\sigma_x + T_z\sigma_z)$$

$$+ P_y(T + \Sigma\sigma_y) - P_z(L_x\sigma_x - L_z\sigma_z)\}, \tag{4}$$

$$\rho_\mathrm{f} = \tfrac{1}{2}(I + \boldsymbol{\sigma}\cdot\boldsymbol{P}_\mathrm{f}), \tag{5}$$

where ρ_f is the density matrix of the recoil nucleon, and $\boldsymbol{P}_\mathrm{f}$ is its polarisation.

Since we can obtain all bilinear products of amplitudes $b_i^*\, b_j$ from the 16 observables given we see that complete information can be obtained without measuring a triple polarisation *. For this reason we do not consider such experiments.

3. Proof

We are now in a position to prove our results. If we measure the set S then we can immediately obtain the moduli $r_i = |b_i|$ of the transversity amplitudes. Now we require a condition that three measurements give us their relative phases, ϕ_i.

The necessity of our condition can be seen trivially from table 1. Let us consider the set BT for definiteness. If we measure three (or indeed four) from this set then we have equations of the form

$$r_1 r_3 \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\}(\phi_1 - \phi_3) \pm r_2 r_4 \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\}(\phi_2 - \phi_4) = K, \tag{6}$$

* This is a special case of the general result proved by Simonius [9].

where K is known. Solving these equations gives $(\phi_1 - \phi_3)$ and $(\phi_2 - \phi_4)$ but *no more*. Therefore our condition is necessary, since the cases BR and TR are similar.

One can see this in another way. Of the eight quantities of sets S and BT only *six* are independent since there are two constraint equations [5]

$$E^2 + F^2 + G^2 + H^2 = 1 + P^2 - \Sigma^2 - T^2, \tag{7}$$

$$FG - EH = P - T\Sigma. \tag{8}$$

To prove the sufficiency is equally straightforward but requires a rather tedious enumeration of possibilities. However it can be shown that any three observables not all of the same group enable the relative phases to be calculated by solving the relevant trigonometric equations. This can easily be seen by inspecting the appropriate Jacobian.

If the three measurements are taken from three different sets then, taking ϕ_i, ϕ_j and ϕ_k as independent variables, the Jacobian has the form

$$J = \begin{vmatrix} T_{ij} & -T_{ij} & T_{kl} \\ T_{ik} & T_{jk} & -T_{ik} \\ T_{il} & T_{jk} & -T_{jk} \end{vmatrix}, \tag{9}$$

where

$$T_{ij} = \pm\, 2r_i r_j \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} (\phi_i - \phi_j), \tag{10}$$

and i, j, k and l are all different. It is obvious from the position of the trigonometric functions of ϕ_l that the determinant is non-zero.

In the case where two of the measurements are from the same set the Jacobian is

$$J = \begin{vmatrix} T_{ij} & -T_{ij} & T_{kl} \\ T'_{ij} & -T'_{ij} & T'_{kl} \\ T_{il} & T_{jk} & -T_{jk} \end{vmatrix}, \tag{11}$$

where T'_{ij} is also of the form given in eq. (10). By realising that if $T_{ij} = T'_{ij}$ then $T_{kl} = -T'_{kl}$ and *vice versa* the determinant can be easily shown to be non-zero. This completes the proof.

An interesting consequence of this result is that in order to obtain complete information one must use a polarised target, polarised beam and measure a recoil polarisation *all* in double polarisation experiments. However one can avoid measuring any one of them in a single polarisation measurement by doing the appropriate double polarisation experiment out of the scattering plane. In fact doing all possible experiments with beam and/or target polarised perpendicular to the scattering plane (or not polarised at all) and measuring the component of recoil polarisation out of the scattering plane is equivalent to measuring the set S. This is well-known property of transversity amplitudes.

Our results disagree with those of Goldstein et al. [4]. We feel that their approach which involves listing the constraints satisfied by the observables is rather dangerous in that one can easily overlook a constraint. This is especially so in the case of constraints on the amplitude phases which appear to be highly non-linear and complicated when written in terms of measurable quantities.

4. Quadrant ambiguities

If we carry out three double polarisation measurements according to our prescription then in general we are left with a discrete eightfold ambiguity. This arises because we are solving three trigonometric equations and each has a twofold ambiguity in its solution. To determine the amplitudes uniquely (up to an overall phase, of course) then we must carry out further measurements. We have proved the following rather surprising result.

In order to determine the amplitudes uniquely then one must make five double polarisation measurements in all, provided that no four of them come from the same set.

Our proof of this statement involves another tedious though elementary enumeration of possibilities, and we do not give it here. However we give an example which we feel illustrates the principles involved. The case we consider is where we have satisfied the criterion of sect. 3 by measuring G, F and L_x. We can now solve for $(\phi_1 - \phi_3)$, $(\phi_2 - \phi_4)$ and $(\phi_1 - \phi_2)$ obtaining the solutions

$$\phi_{13} = \alpha_{13} \quad \text{or} \quad \pi - \alpha_{13},$$

$$\phi_{24} = \alpha_{24} \quad \text{or} \quad \pi - \alpha_{24},$$

$$\phi_{12} - \beta = \gamma \quad \text{or} \quad \pi - \gamma,$$

where we have defined

$$\phi_{ij} = \phi_i - \phi_j, \tag{12}$$

$$\sin(\alpha_{13}) = \frac{1}{2}\frac{G+F}{r_1 r_3}, \tag{13}$$

$$\sin(\alpha_{24}) = \frac{1}{2}\frac{G-F}{r_2 r_4}, \tag{14}$$

$$\tan(\beta) = \frac{r_3 r_4 \sin(\phi_{13} - \phi_{24})}{r_1 r_2 + r_3 r_4 \cos(\phi_{13} - \phi_{24})}, \tag{15}$$

$$\sin(\gamma) = \frac{\frac{1}{2} L_x}{[r_1^2 r_2^2 + r_3^3 r_4^2 + 2 r_1 r_2 r_3 r_4 \cos(\phi_{13} - \phi_{24})]}. \tag{16}$$

If we now measure $E = -2\,[r_1 r_3 \cos(\phi_{13}) + r_2 r_4 \cos(\phi_{24})]$ then we solve both the ϕ_{13} *and* the ϕ_{24} ambiguities since all four choices give different predictions for E in general. This leaves us with an ambiguity between

$$\phi_{12} = \gamma + \beta \quad \text{or} \quad \phi_{12} = \pi - \gamma + \beta, \tag{17}$$

where γ and β are now known unambiguously. We must measure any other observable except H, which provides no further information since it depends only on ϕ_{13} and ϕ_{24}. Another way to see this is to note that if we know T, P, Σ, F, G and E then H is given uniquely by eq. (8). To see that any other measurement will resolve the ambiguity one must examine all the possibilities. L_z measures $\cos(\phi_{12} - \beta)$ and obviously resolves the ambiguity. T_x and T_z measure $\cos(\phi_{12} - \beta')$ and $\sin(\phi_{12} - \beta')$ where

$$\tan(\beta') = \frac{r_3 r_4 \sin(\phi_{13} - \phi_{24})}{r_1 r_2 - r_3 r_4 \cos(\phi_{13} - \phi_{24})}. \tag{18}$$

Therefore if we measure (say) T_z we have

$$\phi_{12} = \gamma' + \beta' \quad \text{or} \quad \pi - \gamma' + \beta', \tag{19}$$

where

$$\sin(\gamma') = \frac{\frac{1}{2} T_z}{r_1^2 r_2^2 + r_3^2 r_4^2 - 2 r_1 r_2 r_3 r_4 \cos(\phi_{13} - \phi_{24})}, \tag{20}$$

and since $\gamma' \neq \gamma$ and $\beta' \neq \beta$ in general* only one solution of eq. (19) will coincide with that of eq. (17) and so the ambiguity is resolved.

Measuring an observable from the set BR would give $\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\{\phi_{12} - \beta''\}$ where β'' is yet another function of ϕ_{13} and ϕ_{24} and so the ambiguity is resolved as before.

It is easy to see that all cases can be solved in exactly the same way, and that one obtains unique amplitudes by measuring any five double polarisation observables, provided four are not from the same set. The complete proof of this is exceedingly tedious and we do not give it here. We have included the above example as an illustration of the method of proof.

Our results differ from those of Goldstein et al. [4] who claim that three measurements are necessary to solve the ambiguities in addition to the seven necessary to obtain the amplitudes up to the ambiguities. This is perhaps what one would naively expect as we have three twofold ambiguities. However, they do not give a proof or even an example of this, and, as we have seen, one measurement can resolve *two* twofold ambiguities.

* It is of course possible that there is an accidental equality between β and β' and then the ambiguity would not be resolved.

5. Discussion

We conclude with a few remarks on experiments concerning particular reactions.

(a) $\gamma p \to \pi^0 p$. Here it is difficult to measure a recoil polarisation since it involves a rescattering. In obtaining P this can be avoided (and has been) by measuring $\{L(\frac{1}{2}\pi, 0); y; 0\}$. Thus one can obtain $d\sigma/dt$, Σ, T, P, G and H* without measuring a recoil. To obtain a complete set one must measure any component of recoil polarisation in the xz plane with either a target polarised in the xz plane *or* a beam either circularly polarised or linear polarised at an angle $\frac{1}{4}\pi$ to the reaction plane and any such** measurement would suffice. However the component of recoil polarisation in the z' direction is not measurable by rescattering.

(b) $\gamma p \to K^+\Lambda$. In this case the decay of the Λ gives all three components of recoil polarisation. Thus in this case one would avoid the set BT and do any three measurements from TR and BR though not all from the same set.

(c) $\gamma p \to \pi^+ n$. Since some measurement of recoil polarisation in a double polarisation experiment is required, it would appear that the difficulty of rescattering experiments for neutrons would rule out the possibility of amplitude analysis in this reaction.

(d) $\gamma n \to \pi^- p$. This is same as (a) with the additional difficulties of using a polarised deuterium target.

Any other practical case can be studied by a simple application of our rules.

The above discussion shows which experiments are in principle necessary to perform an amplitude analysis. However not all such experiments are equally practicable

There may be difficulties due to the smallness of the asymmetry that is to be measured. In this case bounds can be very useful in deciding which experiment to do. The subject of bounds has been studied in some detail by Goldstein et al. [4]. In terms of the classes we have defined the bounds have a certain symmetry. There are the following bounds within the set S.

$$|P \pm T| \leqslant 1 \pm \Sigma, \qquad |T \pm \Sigma| \leqslant 1 \pm P, \qquad |P \pm \Sigma| \leqslant 1 \pm T. \tag{21}$$

Also all double polarisation observables are bounded by the set S as follows

$$|X_{BT}| \leqslant \min\{\sqrt{1-\Sigma^2}, \sqrt{1-T^2}\},$$

where

$$X_{BT} = G, H, E \quad \text{or} \quad F;$$

$$|X_{BR}| \leqslant \min\{\sqrt{1-\Sigma^2}, \sqrt{1-P^2}\},$$

* Or E and F with a circularly polarised beam.

** Any measurement from set TR or BR.

where

$$X_{\mathrm{BR}} = O_x, O_z, C_x \quad \text{or} \quad C_z;$$

$$|X_{\mathrm{TR}}| \leqslant \min\{\sqrt{1-P^2}, \sqrt{1-T^2}\},$$

where

$$X_{\mathrm{TR}} = T_x, T_z, L_x \quad \text{or} \quad L_z. \tag{22}$$

In addition, if one double polarisation has already been measured then the following more stringent bounds between two observables of a given set and the set S, are useful.

$$\begin{aligned}
&\max\{(G^2+E^2), (H^2+F^2), (G^2+H^2), (E^2+F^2)\} \leqslant \min\{(1-\Sigma^2), (1-T^2)\},\\
&\max\{(O_x^2+O_z^2), (C_x^2+C_z^2), (O_x^2+C_x^2), (O_z^2+C_z^2)\} \leqslant \min\{(1-\Sigma^2), (1-P^2)\},\\
&\max\{(T_x^2+T_z^2), (L_x^2+L_z^2), (T_x^2+L_x^2), (T_z^2+L_z^2)\} \leqslant \min\{(1-P^2), (1-T^2)\}.
\end{aligned} \tag{23}$$

$$\begin{aligned}
&\max\{|G \pm F|, |E \pm H|\} \leqslant 1 \pm P,\\
&\max\{|T_z \pm L_x|, |T_x \pm L_z|\} \leqslant 1 \pm \Sigma,\\
&\max\{|O_x \pm C_z|, |O_z \mp C_x|\} \leqslant 1 \pm T.
\end{aligned} \tag{24}$$

These bounds (and many more) are easy to prove, particularly using transversity amplitudes. To our knowledge the bounds (23) have not been noted before.

These bounds can be very useful if a particular asymmetry is near 1. For example, at high energies the four reactions discussed above all have the polarised beam asymmetry Σ near 1 and so quantities bounded by $1-\Sigma$ or $\sqrt{1-\Sigma^2}$ are very restricted.

To resolve quadrant ambiguities our rules indicate which measurements are sufficient *in principle.* However in practice the possibility of accidental degeneracy (or near degeneracy) of predictions of the different solutions might make a choice difficult. This might happen if a quantity is severely bounded by quantities already measured. The easiest way to resolve this is to predict all the observables for each solution and to see which measurements allow the clearest choice between solutions.

Finally we remark that although the problem of obtaining the seven measurements needed to determine the amplitudes if not all measurements of the set S are included is of only mathematical interest nevertheless sufficiency conditions can be determined by a generalisation of our method. This involves selecting a set S′ of four measurements required to determine the moduli of a new set of amplitudes. For instance if we use helicity amplitudes than the set S′ consists of $\mathrm{d}\sigma/\mathrm{d}t$, E, C_z and L_z, and we can then divide the remaining twelve measurements into three classes of four according to which amplitudes interfere. Our results can then be trivially generalised.

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