Photon and Electron Polarization in High-Energy Bremsstrahlung and Pair Production with Screening*

HAAKON OLSEN, *Fysisk Institutt, Norges Tekniske Høgskole, Trondheim, Norway*

AND

L. C. MAXIMON,† *Fysisk Institutt, Norges Tekniske Høgskole, Trondheim, Norway and Department of Theoretical Physics, The University, Manchester, England*

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The matrix element for bremsstrahlung and pair production is written in a particularly simple form which reduces to the well-known matrix element for nonrelativistic energies, in terms of a vector which is closely related to the nonrelativistic current density vector.

The cross section for high-energy bremsstrahlung and pair production involving arbitrarily polarized photons and electrons has been calculated. The Coulomb and screening effects are taken into account exactly. It is found that the screening and Coulomb corrections to the polarization-dependent part of the cross section are analytically of the same form and numerically of the same magnitude as the corrections to the polarization-independent part of the cross section found earlier. We also give the cross section for bremsstrahlung summed over spin and polarization directions and integrated over the direction of motion of the final particle, i.e., the angular distribution of the radiation, which has a very simple form.

Formulas are given for the linear and circular polarization of bremsstrahlung from arbitrarily polarized electrons and for the spin polarization of pairs from arbitrarily polarized photons. The circular polarization of bremsstrahlung from longitudinally polarized electrons is complete at the upper end of the spectrum and is much larger than from transversely polarized electrons throughout the spectrum. In the same way, circularly polarized photons produce longitudinally polarized electron-positron pairs, the probability of producing transversely polarized pair particles being, in general, much smaller. The faster one of the pair particles is always polarized in the same sense as the photon. On the other hand, the linear polarization of bremsstrahlung is smallest at the upper end of the spectrum and increases with decreasing photon energy. It is independent of the spin of the initial electron if one sums over the spin of the final particle. The circular polarization of the emitted photons is to a high degree independent of screening and Coulomb corrections, at the high energies considered here. The linear polarization is, however, significantly dependent on these corrections. Similar conclusions hold for pair production from circularly and linearly polarized photons, respectively.

Finally, the electron spin-photon polarization correlation for bremsstrahlung and pair production is discussed. The depolarization, because of bremsstrahlung, of polarized electrons passing through matter is calculated and is appreciable over a radiation length. It is greater for a transversely polarized electron than for a longitudinally polarized electron.

1. INTRODUCTION

THE linear polarization of bremsstrahlung was studied first by Sommerfeld[1] for low electron energy and by May and Wick[2] for relativistic energies. It was found that for low photon energies the electric vector of the radiation is in the plane of emission, which is what would be expected from classical considerations. The photons of high energy, on the other hand, are emitted in the plane of the magnetic vector. Thus at high energies the radiation is always polarized in a direction perpendicular to the plane of emission. In addition to the work of May and Wick there have been calculations by Gluckstern *et al.*[3] and by Gluckstern and Hull.[4] These calculations have always been performed in the Born approximation and with exponential screening. The present work supplements these papers at high energies in that it takes into account the Coulomb correction and the screening exactly.

It was apparently noticed first by Zel'dovich[5] that the bremsstrahlung emitted from polarized electrons may be circularly polarized. Since the discovery that the particles emitted in weak processes are polarized, a number of papers on circular polarization of bremsstrahlung have appeared.[6–10] In all these articles the Born approximation has been applied, and the screening has been taken into account by an exponential screening factor. As for the case of linear polarization we extend these results at high energies by taking into account the Coulomb correction and screening exactly. We also give the cross section for definite spins of the final electron; this then allows us to calculate the depolarization of polarized electrons because of emission of bremsstrahlung. A preliminary note of some of these results has already been given.[11,12]

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† Present address: National Bureau of Standards, Washington, D. C.

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[2] G. C. Wick, Phys. Rev. **81**, 467 (1951); M. May and G. C. Wick, Phys. Rev. **81**, 628 (1951); M. M. May, Phys. Rev. **84**, 265 (1951).

[3] Gluckstern, Hull, and Breit, Science **114**, 480 (1951); Phys. Rev. **90**, 1026 (1953).

[4] R. L. Gluckstern and M. H. Hull, Phys. Rev. **90**, 1030 (1953).

[5] Ia. B. Zel'dovich, Doklady Akad. Nauk S.S.S.R. **83**, 63 (1952).

[6] G. Böbel, Nuovo cimento **6**, 1241 (1957).

[7] A. Claesson, Arkiv Fysik **12**, 569 (1957).

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[9] C. Fronsdal and H. Überall, Nuovo cimento **8**, 163 (1958); Phys. Rev. **111**, 580 (1958).

[10] Haridas Banerjee, Phys. Rev. **111**, 532 (1958). The case of arbitrary photon polarization without screening is considered.

[11] Olsen, Wergeland, and Maximon, Bull. Am. Phys. Soc. II, **3**, 174 (1958).

[12] H. Olsen and L. C. Maximon, Phys. Rev. **110**, 589 (1958). It should be noticed that the term

$$+[(\epsilon_1^2-\epsilon_2^2)(3+2\Gamma)-2k\epsilon_2(1+4u^2\xi^2\Gamma)]\mathbf{s}_1\cdot\hat{\mathbf{k}}(i\hat{\mathbf{k}}\cdot\boldsymbol{\lambda}^*\times\boldsymbol{\lambda})$$

in Eq. (1) of this reference should have the opposite sign; see Eq. (7.1) in the present paper.

In Sec. 2 we discuss a simple method for obtaining the cross section directly in terms of the initial and final spins of the electron. The differential cross section for bremsstrahlung with specified initial and final spins of the electron and specified polarization of the emitted spectrum is then derived in Secs. 3 and 4. In Secs. 5 and 6 we integrate over the direction of motion of the final electron, for the cases of no screening, intermediate screening, and complete screening. Formulas for linear and circular polarization of bremsstrahlung are given in Sec. 7, and in Sec. 8 we discuss the correlations between the spins of the initial and final electron and the photon polarization vector. In particular the circular polarization of the photon beam is considered. The discussion covers all cases of screening. The depolarization of the polarized electrons in the radiation process is derived in Sec. 9. Finally, in Sec. 10 we write down the corresponding quantities for pair production.

2. SPIN FORMALISM

We use a method by which the absolute square of the matrix element appears directly in terms of the spin vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ of the initial and final states, respectively.[13] The procedure is simplified by the fact that the spin part of the wave function at high energies involves a plane-wave spinor u. It has in fact been shown[14] that for processes such as high-energy bremsstrahlung and pair production in which the momentum transfer q is always much smaller than the momentum p of the electron, i.e., processes in which significant angular momenta are $l\gg 1$, the Sommerfeld-Maue type wave function,

$$\psi_{\pm}=e^{i\mathbf{p}\cdot\mathbf{r}}\left(1-\frac{i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}}{2\epsilon}\right)uF_{\pm}, \tag{2.1}$$

is a good approximation. $\mathbf{p}$ is the momentum of the electron in units of mc, $\epsilon=(p^2+1)^{\frac{1}{2}}$ its energy in units of mc^2, $\boldsymbol{\alpha}$ the Dirac operator, and $\mathbf{r}$ the electron's coordinate in units of $\hbar/mc$. u is the free-particle spinor. $F_{\pm}$ is the solution of

$$(\nabla^2+2i\mathbf{p}\cdot\boldsymbol{\nabla}-2\epsilon V)F=0, \tag{2.2}$$

normalized so that $F(r)\rightarrow 1$ as $r\rightarrow\infty$. The subscript $+$ $(-)$ refers to wave functions with the asymptotic form of a plane wave plus outgoing (ingoing) spherical waves. For an unscreened Coulomb potential $V=-a/r$, $a=Ze^2/\hbar c$, the solution F was given by Sommerfeld and Maue,[15] and for an arbitrarily screened potential by Olsen, Maximon, and Wergeland.[16]

[13] The present method differs from that which is usually applied in polarization calculations. See, e.g., F. W. Lipps and H. A. Tolhoek, Physica **20**, 85, 395 (1954); H. A. Tolhoek, Revs. Modern Phys. **28**, 277 (1956).

[14] H. A. Bethe and L. C. Maximon, Phys. Rev. **93**, 768 (1954); in the following referred to as BM.

[15] A. Sommerfeld and A. W. Maue, Ann. Physik **22**, 629 (1935).

[16] Olsen, Maximon, and Wergeland, Phys. Rev. **106**, 27 (1957); in the following referred to as OMW.

The free-particle spinor u satisfies

$$(\boldsymbol{\alpha}\cdot\mathbf{p}+\beta-\epsilon)u=0. \tag{2.3}$$

We want to write u in such a way that the two-component Pauli spinor v appears explicitly. To this end, we use the representation

$$\boldsymbol{\alpha}=\begin{pmatrix}0&\boldsymbol{\sigma}\\ \boldsymbol{\sigma}&0\end{pmatrix}=\rho_1\boldsymbol{\sigma},\quad \beta=\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\rho_3,$$
$$u=N\begin{pmatrix}v\\w\end{pmatrix}, \tag{2.4}$$

where $\boldsymbol{\sigma}$ is the Pauli spin matrix vector and $\boldsymbol{\rho}$ the corresponding Dirac matrix vector[17] in "charge space." v and w are two-component spinors (the "large" and "small" components, respectively), and N a normalization factor. Introducing these expressions into the four-component wave equation (2.3), this equation splits into two two-component equations:

$$\boldsymbol{\sigma}\cdot\mathbf{p}w+(1-\epsilon)v=0,\quad \boldsymbol{\sigma}\cdot\mathbf{p}v-(1+\epsilon)w=0. \tag{2.5}$$

The desired solution is thus

$$u=N\begin{pmatrix}v\\ [\boldsymbol{\sigma}\cdot\mathbf{p}/(\epsilon+1)]v\end{pmatrix}=N\begin{pmatrix}1\\ \boldsymbol{\sigma}\cdot\mathbf{p}/(\epsilon+1)\end{pmatrix}v,$$
$$N=[(\epsilon+1)/2\epsilon]^{\frac{1}{2}}, \tag{2.6}$$

where v has been assumed to be normalized. For applications it should be noted that the separation of the spin space from the charge space is complete in the sense that in (2.6) the matrices ρ_k operate only on the charge space part of u, viz.,

$$\begin{pmatrix}1\\ \boldsymbol{\sigma}\cdot\mathbf{p}/(\epsilon+1)\end{pmatrix}.$$

v is clearly the Pauli spin state referred to the system in which the electron is at rest. Let $\mathbf{s}/s=\boldsymbol{\zeta}$ $(s=\frac{1}{2})$ be the spin direction in this system. Then

$$\boldsymbol{\zeta}\cdot\boldsymbol{\sigma}v=v. \tag{2.7}$$

The well-known explicit solution of this equation is

$$v=\frac{1}{[2(1+\zeta_z)]^{\frac{1}{2}}}\begin{pmatrix}1+\zeta_z\\ \zeta_x+i\zeta_y\end{pmatrix}, \tag{2.8}$$

or in terms of the angles χ, ϕ_s, where

$$\boldsymbol{\zeta}=\{\sin\chi\cos\phi_s,\ \sin\chi\sin\phi_s,\ \cos\chi\},$$

we can write

$$v=\begin{pmatrix}e^{-i\phi_s/2}\cos(\chi/2)\\ e^{i\phi_s/2}\sin(\chi/2)\end{pmatrix}. \tag{2.9}$$

[17] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Clarendon Press, Oxford, 1947), third edition, pp. 255, 256.

We shall not, however, need these explicit forms. In fact, only (2.10) and (2.11) below are needed for the following calculation.

Multiplying the equation adjoint to (2.7) by v from the left, we have

$$vv^{\dagger}(\boldsymbol{\zeta}\cdot\boldsymbol{\sigma}-1)=0.$$

For normalized v the solution is clearly

$$vv^{\dagger}=\tfrac{1}{2}(1+\boldsymbol{\zeta}\cdot\boldsymbol{\sigma}). \tag{2.10}$$

$vv^{\dagger}$ is the projection operator for the spin state v. We shall also need the well-known diagonal matrix elements

$$(v,\boldsymbol{\sigma}v)=\boldsymbol{\zeta}. \tag{2.11}$$

This also follows from (2.7) by multiplication from the left by $v^{\dagger}$ and noting that $\boldsymbol{\zeta}$ is the only vector in the rest system.

The high-energy wave function which we shall use for an electron may thus be written (since in this limit $N=1/\sqrt{2}$)

$$\psi_{\mathrm{el},\pm}=\frac{1}{\sqrt{2}}e^{i\mathbf{p}\cdot\mathbf{r}}\left(1-\frac{i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}}{2\epsilon}\right)\begin{pmatrix}1\\ \boldsymbol{\sigma}\cdot\mathbf{p}/(\epsilon+1)\end{pmatrix}vF_{\pm}. \tag{2.12}$$

In the matrix element for pair production there appears the charge conjugate of the positron wave function having the asymptotic form of a plane wave plus ingoing spherical waves, viz., $C\psi_{\mathrm{pos},-}^{*}(\epsilon,\mathbf{p},\boldsymbol{\zeta})$, where $C=i\beta\alpha_y$. $C\psi_{\mathrm{pos},-}^{*}$ can also be considered as the wave function (not conjugate) of a negatively charged electron with energy $-\epsilon$, momentum $-\mathbf{p}$, $-p$, spin $-\boldsymbol{\zeta}$ with the same asymptotic form of a plane wave plus ingoing spherical waves:

$$C\psi_{\mathrm{pos},-}^{*}(\epsilon,\mathbf{p},p,\boldsymbol{\zeta})=\psi_{\mathrm{el},-}(-\epsilon,-\mathbf{p},-p,-\boldsymbol{\zeta}). \tag{2.13}$$

The spinor factor in the wave function $C\psi_{\mathrm{pos},-}^{*}$ is therefore, from (2.6) and (2.7),

$$u=N\begin{pmatrix}1\\ \boldsymbol{\sigma}\cdot\mathbf{p}/(\epsilon-1)\end{pmatrix}v(-\boldsymbol{\zeta}),\quad N=\left(\frac{\epsilon-1}{2\epsilon}\right)^{\frac{1}{2}}, \tag{2.14}$$

where $v(-\boldsymbol{\zeta})$ is a normalized solution of

$$\boldsymbol{\zeta}\cdot\boldsymbol{\sigma}v=-v. \tag{2.15}$$

Alternatively to (2.13), the *spatial* part of the positron wave function may be obtained directly from that of an electron by reversing the sign of the Coulomb interaction, so that we may also write

$$C\psi_{\mathrm{pos},-}^{*}(\epsilon,\mathbf{p},\boldsymbol{\zeta})=\frac{1}{\sqrt{2}}e^{-i\mathbf{p}\cdot\mathbf{r}}\left(1+\frac{i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}}{2\epsilon}\right)\begin{pmatrix}1\\ \boldsymbol{\sigma}\cdot\mathbf{p}/(\epsilon-1)\end{pmatrix}\times v(-\boldsymbol{\zeta})F_{-}^{*}(-Z), \tag{2.16}$$

i.e.,

$$F_{-}^{*}(\epsilon,\mathbf{p},p,-Z)=F_{-}(-\epsilon,-\mathbf{p},-p,Z).$$

3. MATRIX ELEMENTS FOR BREMSSTRAHLUNG

We calculate the amplitude $\mathbf{A}\cdot\mathbf{e}^{*}$ for bremsstrahlung[18] for specified spins of the initial and final electrons using the representation (2.12) of the wave function, in which the two-component spinor v is separated out. $\mathbf{e}$ is a vector with complex components a_1 and a_2 in the x and y directions, respectively:

$$\mathbf{e}=a_1\mathbf{e}_x+a_2\mathbf{e}_y, \tag{3.1}$$

and is normalized so that

$$|\mathbf{e}|^2=|a_1|^2+|a_2|^2=1. \tag{3.2}$$

By choosing a_1 and a_2 appropriately, one obtains the amplitude for any desired polarization. Thus the amplitude for radiation plane polarized in the x direction is found by choosing $a_1=1$ and $a_2=0$; in the y direction by choosing $a_1=0$, $a_2=1$. The amplitude for circularly polarized radiation is obtained by setting $a_1=1/\sqrt{2}$ and $a_2=\pm i/\sqrt{2}$, the upper and lower sign referring to right and left circular polarization, respectively. (We use the same convention as in Blatt and Weisskopf.[19])

The amplitude[20] is given by

$$\mathbf{A}\cdot\mathbf{e}^{*}=(\psi_{2,-},\ \boldsymbol{\alpha}\cdot\mathbf{e}^{*}e^{-i\mathbf{k}\cdot\mathbf{r}}\psi_{1,+}) \tag{3.3}$$

with $\psi_{1,+}$ and $\psi_{2,-}$ as given in (2.12). Neglecting terms of relative order $1/\epsilon$, we have

$$\mathbf{A}\cdot\mathbf{e}^{*}=(u_2,\ \{\boldsymbol{\alpha}\cdot\mathbf{e}^{*}I_1+\boldsymbol{\alpha}\cdot\mathbf{e}^{*}\boldsymbol{\alpha}\cdot\mathbf{I}_2+\boldsymbol{\alpha}\cdot\mathbf{I}_3\boldsymbol{\alpha}\cdot\mathbf{e}^{*}\}u_1), \tag{3.4}$$

where I_1, $\mathbf{I}_2$, $\mathbf{I}_3$, defined before,[21] are given by

$$I_1=\int F_{2,-}^{*}e^{i\mathbf{q}\cdot\mathbf{r}}F_{1,+}d^3r,$$

$$\mathbf{I}_2=-\frac{i}{2\epsilon_1}\int F_{2,-}^{*}e^{i\mathbf{q}\cdot\mathbf{r}}\boldsymbol{\nabla}F_{1,+}d^3r,\quad \text{(Brems.)} \tag{3.5}$$

$$\mathbf{I}_3=\frac{i}{2\epsilon_2}\int(\boldsymbol{\nabla}F_{2,-}^{*})e^{i\mathbf{q}\cdot\mathbf{r}}F_{1,+}d^3r.$$

These integrals have been evaluated for an unscreened Coulomb potential (BM),[14] and also for the case of arbitrary screening (OMW).[16]

For the following calculation it will be useful to note that the three integrals (3.5) are related. By a

[18] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, New York, 1954), third edition, p. 143. It should be noted that in Eq. (21b) of this reference $\boldsymbol{\alpha}\cdot\mathbf{e}^{*}$ should appear in place of $\boldsymbol{\alpha}\cdot\mathbf{e}$, as is apparent from Eq. (19).

[19] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952); note p. 807, Eq. (5.1).

[20] Reference 18. In order to simplify the following equations, numerical factors are omitted in this definition of the amplitude. These are finally included in the expression for the matrix element for bremsstrahlung, Eq. (4.1) in the present paper.

[21] Reference 16, Eq. (4.2).

partial integration on $\mathbf{I}_3$, one finds

$$\mathbf{I}_3=-\frac{i}{2\epsilon_2}\int F_{2,-}{}^*\nabla(e^{i\mathbf{q}\cdot\mathbf{r}}F_{1,+})d^3r$$
$$=-\frac{i}{2\epsilon_2}\int F_{2,-}{}^*e^{i\mathbf{q}\cdot\mathbf{r}}\nabla F_{1,+}d^3r+\frac{\mathbf{q}}{2\epsilon_2}\int F_{2,-}{}^*e^{i\mathbf{q}\cdot\mathbf{r}}F_{1,+}d^3r.$$

Therefore, comparing with the definitions (3.5), we have

$$\mathbf{I}_3=\frac{\epsilon_1}{\epsilon_2}\mathbf{I}_2+\frac{\mathbf{q}}{2\epsilon_2}I_1. \tag{3.6}$$

The amplitude $\mathbf{A}\cdot\mathbf{e}^*$ may thus be expressed in terms of only two of the integrals (3.5). Noting that at high energies only the components of $\mathbf{I}_2$ and $\mathbf{I}_3$ perpendicular to $\mathbf{k}$ contribute to the matrix element [see Eqs. (3.4), (3.13)–(3.15)], it will prove convenient to introduce the vector $\mathbf{J}$ given by

$$J_z=-\frac{1}{2\epsilon_1\epsilon_2}I_1, \qquad \text{(Brems.)} \quad (3.7)$$
$$\mathbf{J}_\perp=\frac{\mathbf{u}}{2\epsilon_1\epsilon_2}I_1+\frac{1}{\epsilon_2}\mathbf{I}_{2\perp}=\frac{\mathbf{v}}{2\epsilon_1\epsilon_2}I_1+\frac{1}{\epsilon_1}\mathbf{I}_{3\perp}.$$

The latter equality results because of (3.6), noting that $\mathbf{q}_\perp=\mathbf{u}-\mathbf{v}$. $\mathbf{u}$ and $\mathbf{v}$ are the components of $\mathbf{p}_1$ and $\mathbf{p}_2$ perpendicular to $\mathbf{k}$, respectively. Using Eqs. (4.2) and (7b.10) of OMW, the vector $\mathbf{J}$ is given by

$$\mathbf{J}=B\{\mathbf{u}\xi-\mathbf{v}\eta+\hat{k}(\xi-\eta)\}, \quad \text{(Brems.)} \quad (3.8)$$

with

$$\xi=(1+u^2)^{-1},\quad \eta=(1+v^2)^{-1},$$
$$u=p_1\theta_1,\quad v=p_2\theta_2, \tag{3.8a}$$
$$B=(4\pi a/kq^2)A,\quad \hat{k}=\mathbf{k}/k,$$

where A is given for arbitrary screening by Eq. (7b.10) of OMW. For the special case of no screening, one has [Eq. (7b.11) of OMW and Eq. (8.20) of BM],

$$|B|^2=(4\pi a/kq^2)^2R(y),$$
$$R(y)=[V^2(x)+a^2y^2W^2(x)]/V^2(1), \tag{3.9}$$
$$y\equiv1-x=\delta^2/(\xi\eta q^2),\quad \delta=q_{\min}=k/(2\epsilon_1\epsilon_2).$$

V and W are the hypergeometric functions

$$V(x)=F(ia,-ia;1;x),$$
$$W(x)=F(1+ia,1-ia;2;x), \tag{3.9a}$$
$$V(1)=F(ia,-ia;1;1)=|\Gamma(1+ia)|^{-2}=\sinh\pi a/\pi a.$$

We now turn to the amplitude $\mathbf{A}\cdot\mathbf{e}^*$ given in (3.4). Written in terms of the spin amplitudes v_1 and v_2, the first term becomes

$$(u_2,\boldsymbol{\alpha}\cdot\mathbf{e}^*u_1)I_1$$
$$=\frac{1}{2}\left(v_2,\left(1\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_2}{\epsilon_2+1}\right)\rho_1\boldsymbol{\sigma}\cdot\mathbf{e}^*\left(\frac{1}{\boldsymbol{\sigma}\cdot\mathbf{p}_1/(\epsilon_1+1)}\right)v_1\right)I_1$$
$$=\frac{1}{2}\left(v_2,\left\{\boldsymbol{\sigma}\cdot\mathbf{e}^*\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1}{\epsilon_1+1}+\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_2}{\epsilon_2+1}\boldsymbol{\sigma}\cdot\mathbf{e}^*\right\}v_1\right)I_1.$$

Separating the components of $\mathbf{p}_1$ and $\mathbf{p}_2$ in the direction of $\mathbf{k}$ and perpendicular to $\mathbf{k}$, we have

$$\boldsymbol{\sigma}\cdot\mathbf{e}^*\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1}{\epsilon_1+1}+\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_2}{\epsilon_2+1}\boldsymbol{\sigma}\cdot\mathbf{e}^*=\boldsymbol{\sigma}\cdot\mathbf{e}^*\frac{\boldsymbol{\sigma}\cdot\mathbf{u}}{\epsilon_1+1}+\frac{\boldsymbol{\sigma}\cdot\mathbf{v}}{\epsilon_2+1}\boldsymbol{\sigma}\cdot\mathbf{e}^*$$
$$+\boldsymbol{\sigma}\cdot\mathbf{e}^*\sigma_z\left(\frac{p_{1z}}{\epsilon_1+1}-\frac{p_{2z}}{\epsilon_2+1}\right). \tag{3.10}$$

Neglecting terms of relative order $1/\epsilon$, we may therefore write

$$(u_2,\boldsymbol{\alpha}\cdot\mathbf{e}^*u_1)I_1=\left(v_2,\left\{\frac{\boldsymbol{\sigma}\cdot\mathbf{e}^*}{2\epsilon_1}(\boldsymbol{\sigma}\cdot\mathbf{u}-\sigma_z)I_1+(\boldsymbol{\sigma}\cdot\mathbf{v}-\sigma_z)I_1\frac{\boldsymbol{\sigma}\cdot\mathbf{e}^*}{2\epsilon_2}\right\}v_1\right). \tag{3.11}$$

We shall use this form later in Eq. (3.15) for the amplitude.

The second term in $\mathbf{A}\cdot\mathbf{e}^*$ is

$$(u_2,\boldsymbol{\alpha}\cdot\mathbf{e}^*\boldsymbol{\alpha}\cdot\mathbf{I}_2u_1)$$
$$=\frac{1}{2}\left(v_2,\left(1\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_2}{\epsilon_2+1}\right)\boldsymbol{\sigma}\cdot\mathbf{e}^*\boldsymbol{\sigma}\cdot\mathbf{I}_2\left(\frac{1}{\boldsymbol{\sigma}\cdot\mathbf{p}_1/(\epsilon_1+1)}\right)v_1\right)$$
$$=\frac{1}{2}\left(v_2,\left\{\boldsymbol{\sigma}\cdot\mathbf{e}^*\boldsymbol{\sigma}\cdot\mathbf{I}_2+\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_2}{\epsilon_2+1}\boldsymbol{\sigma}\cdot\mathbf{e}^*\boldsymbol{\sigma}\cdot\mathbf{I}_2\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1}{\epsilon_1+1}\right\}v_1\right). \tag{3.12}$$

Now

$$\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_2}{\epsilon_2+1}\approx\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1}{\epsilon_1+1}=\sigma_z+O(1/\epsilon).$$

Therefore the part of the last term in (3.12) containing σ_zI_{2z} exactly cancels the part from the first term containing σ_zI_{2z}, while the part containing $\boldsymbol{\sigma}_\perp\cdot\mathbf{I}_{2\perp}$ adds to the first term, giving $2\boldsymbol{\sigma}\cdot\mathbf{e}^*\boldsymbol{\sigma}\cdot\mathbf{I}_{2\perp}$. The amplitude $\mathbf{A}\cdot\mathbf{e}^*$ thus depends only on the perpendicular components $\mathbf{I}_{2\perp}$ and $\mathbf{I}_{3\perp}$:

$$(u_2,\boldsymbol{\alpha}\cdot\mathbf{e}^*\boldsymbol{\alpha}\cdot\mathbf{I}_2u_1)=(v_2,\boldsymbol{\sigma}\cdot\mathbf{e}^*\boldsymbol{\sigma}\cdot\mathbf{I}_{2\perp}v_1), \tag{3.13}$$

and in exactly the same way we find

$$(u_2,\boldsymbol{\alpha}\cdot\mathbf{I}_3\boldsymbol{\alpha}\cdot\mathbf{e}^*u_1)=(v_2,\boldsymbol{\sigma}\cdot\mathbf{I}_{3\perp}\boldsymbol{\sigma}\cdot\mathbf{e}^*v_1). \tag{3.14}$$

For the amplitude we may then write, recalling (3.7),

$$\mathbf{A}\cdot\mathbf{e}^*=\Big(v_2,\ \Big\{\boldsymbol{\sigma}\cdot\mathbf{e}^*\Big[\frac{\boldsymbol{\sigma}\cdot\mathbf{u}}{2\epsilon_1}I_1+\boldsymbol{\sigma}\cdot\mathbf{I}_{2\perp}-\frac{\sigma_z}{2\epsilon_1}I_1\Big]+\Big[\frac{\boldsymbol{\sigma}\cdot\mathbf{v}}{2\epsilon_2}I_1+\boldsymbol{\sigma}\cdot\mathbf{I}_{3\perp}-\frac{\sigma_z}{2\epsilon_2}I_1\Big]\boldsymbol{\sigma}\cdot\mathbf{e}^*\Big\}v_1\Big) \quad (3.15)$$

$$=(v_2,\ \{\epsilon_2\boldsymbol{\sigma}\cdot\mathbf{e}^*\boldsymbol{\sigma}\cdot\mathbf{J}+\epsilon_1\boldsymbol{\sigma}\cdot\mathbf{J}\boldsymbol{\sigma}\cdot\mathbf{e}^*\}v_1).$$

The matrix element thus depends on the integrals (3.5) only through the vector $\mathbf{J}$ of Eq. (3.7). Using the rule

$$\boldsymbol{\sigma}\cdot\mathbf{J}\boldsymbol{\sigma}\cdot\mathbf{e}^*=\mathbf{J}\cdot\mathbf{e}^*+i\boldsymbol{\sigma}\cdot\mathbf{J}\times\mathbf{e}^*, \quad (3.16)$$

we have the simple expression for the amplitude, for arbitrarily polarized bremsstrahlung,

$$\mathbf{A}\cdot\mathbf{e}^*=(v_2,\{(\epsilon_1+\epsilon_2)\mathbf{J}\cdot\mathbf{e}^*+ik\boldsymbol{\sigma}\times\mathbf{J}\cdot\mathbf{e}^*\}v_1). \quad \text{(Brems.)}\ (3.17)$$

The corresponding expression for pair production is obtained from (3.15) and (3.17) by making the substitutions ϵ_1, k, $\mathbf{e}^*\rightarrow-\epsilon_1$, $-k$, $\mathbf{e}$ and by replacing the bremsstrahlung expressions $\mathbf{J}_\perp$ and J_z by those for pair production. This substitution is justified by the discussion at the beginning of Sec. 5, where it is noted that the outgoing or ingoing nature of the wavefunction is contained only in F [Eq. (2.1)], and not in the free-particle spinor u, with which the substitution $\epsilon_1\rightarrow-\epsilon_1$ is concerned. For pair production, the $\mathbf{J}_\perp$ and J_z analogous to (3.7) are

$$J_z=\frac{1}{2\epsilon_1\epsilon_2}I_1, \quad \text{(P.P.)}\ (3.18)$$

$$\mathbf{J}_\perp=\frac{\mathbf{u}}{2\epsilon_1\epsilon_2}I_1+\frac{1}{\epsilon_2}\mathbf{I}_{2\perp}=-\frac{\mathbf{v}}{2\epsilon_1\epsilon_2}I_1-\frac{1}{\epsilon_1}\mathbf{I}_{3\perp},$$

where now the integrals (3.5) are replaced by

$$I_1=\int F_{2,-}{}^*e^{i\mathbf{q}\cdot\mathbf{r}}F_{1,-}d^3r,$$

$$\mathbf{I}_2=\frac{i}{2\epsilon_1}\int F_{2,-}{}^*e^{i\mathbf{q}\cdot\mathbf{r}}\boldsymbol{\nabla}F_{1,-}d^3r, \quad \text{(P.P.)}\ (3.19)$$

$$\mathbf{I}_3=\frac{i}{2\epsilon_2}\int(\boldsymbol{\nabla}F_{2,-}{}^*)e^{i\mathbf{q}\cdot\mathbf{r}}F_{1,-}d^3r.$$

For pair production, for large values of the momentum transfer q, screening is unimportant and $\mathbf{J}$ is given by BM Eq. (6.23) or OMW Eq. 6b.5) and the equations following OMW Eq. (6b.5):

$$\mathbf{J}_\perp=\frac{4\pi a}{kV(1)}\Big\{\frac{1}{q^2}(\mathbf{u}\xi+\mathbf{v}\eta)V(x)+ia\xi\eta(\mathbf{u}\xi-\mathbf{v}\eta)W(x)\Big\}, \quad \text{(P.P.)}\ (3.20)$$

$$J_z=\frac{4\pi a}{kV(1)}\Big\{\frac{1}{q^2}(\xi-\eta)V(x)+ia\xi\eta(\xi+\eta-1)W(x)\Big\}.$$

In (3.20) the quantities $\mathbf{u}$, $\mathbf{v}$, ξ, and η are the same as in (3.8a). However, x and y are given by

$$y\equiv1-x=\xi\eta q^2, \quad \text{(P.P.)}\ (3.20a)$$

rather than as in (3.9). V and W in (3.20) are the functions defined in (3.9a), but have as argument x as defined in (3.20a). For small q, where the screening is effective, the Coulomb correction is negligible. Thus in this case $\mathbf{J}$ is given by the Born-approximation value including screening, *viz.*, from OMW Eqs. (7a.5)–(7a.10) and the equations immediately following (7b.11),

$$\mathbf{J}_{\text{Born}}=\frac{4\pi a}{k}\{(\mathbf{u}\xi+\mathbf{v}\eta)+\hat{k}(\xi-\eta)\}\frac{\{1-F(q)\}}{q^2}, \quad \text{(P.P.)}\ (3.21)$$

$F(q)$ being the atom form factor.[22]

It may be noted that with

$$\Psi=e^{i\mathbf{p}\cdot\mathbf{r}}F, \quad (3.22)$$

we may write $\mathbf{J}_\perp$ and J_z in the form

$$\mathbf{J}_\perp=\frac{1}{2\epsilon_1\epsilon_2}\int\mathbf{j}_\perp e^{-i\mathbf{k}\cdot\mathbf{r}}d^3r, \quad (3.23)$$

$$J_z=-\frac{1}{2\epsilon_1\epsilon_2}\int\rho e^{-i\mathbf{k}\cdot\mathbf{r}}d^3r, \quad (3.24)$$

with

$$\mathbf{j}=\frac{1}{2i}(\Psi_2{}^*\boldsymbol{\nabla}\Psi_1-\Psi_1\boldsymbol{\nabla}\Psi_2{}^*),\quad \rho=\Psi_2{}^*\Psi_1, \quad (3.25)$$

as may be verified by comparison with (3.7) after performing a partial integration. Now in the nonrelativistic limit (in which $k\rightarrow0$), the amplitude (3.17) [apart from the trivial spin factor (v_2,v_1)] approaches

$$\mathbf{A}\cdot\mathbf{e}^*=2\mathbf{J}_\perp\cdot\mathbf{e}^*=\int\mathbf{j}_\perp\cdot\mathbf{e}^*e^{-i\mathbf{k}\cdot\mathbf{r}}d^3r. \quad (3.26)$$

[That $\mathbf{A}\cdot\mathbf{e}^*$ as given in (3.17) approaches the nonrelativistic amplitude $2\mathbf{J}_\perp\cdot\mathbf{e}^*$ is of course accidental, since

[22] Reference 16, equation preceding (7b.12).

in the derivation of (3.17) we have made the high-energy approximation throughout. It may be noted, however, that this same limit is obtained for $\mathbf{A}\cdot\mathbf{e}^*$ as defined in (3.4), using (3.6) and observing that in (2.6), in the low-energy limit, $N=1$.] Thus, since Ψ is, in the nonrelativistic limit, the exact solution to the Schrödinger equation, the amplitude given in (3.17) for relativistically high-energy bremsstrahlung is also correct in the nonrelativistic limit. The spin-dependent term in (3.17) is, apart from the energy factors, that which would be expected from a radiating magnetic dipole of moment $\boldsymbol{\sigma}$ with $\mathbf{J}=\int \mathbf{j}e^{-i\mathbf{k}\cdot\mathbf{r}}d^3r$. However, since $\int \rho e^{-i\mathbf{k}\cdot\mathbf{r}}d^3r=\int j_z e^{-i\mathbf{k}\cdot\mathbf{r}}d^3r$, the vector $\mathbf{J}$ occurring in (3.7) and (3.17) has the opposite sign on its z component to that which would make such an explanation of this term plausible.

4. DIFFERENTIAL CROSS SECTION

In terms of the amplitude $\mathbf{A}\cdot\mathbf{e}^*$ given in (3.3), the matrix element H_{12}' for bremsstrahlung is[18,20]

$$H_{12}'=-e\hbar c(2\pi/k)^{\frac{1}{2}}(\hbar/mc)^3\mathbf{A}\cdot\mathbf{e}^*. \qquad (4.1)$$

The transition probability per unit time is

$$w=(2\pi/\hbar)\rho_f|H_{12}'|^2, \qquad (4.2)$$

in which the density of final states ρ_f is

$$\rho_f=(mc^2)^4(2\pi\hbar c)^{-6}k^2dkd\Omega_1p_2\epsilon_2d\Omega_2. \qquad (4.3)$$

The differential cross section, $d\sigma$, is equal to the transition probability normalized to unit current of the incident particle. We therefore divide w by the velocity of the incident electron, cp_1/ϵ_1, and obtain (after making the high-energy approximation $\epsilon=p$) the differential cross section for polarized bremsstrahlung:

$$d\sigma=\frac{1}{(2\pi)^4}\frac{e^2}{mc^2}\frac{\hbar}{mc}\frac{\epsilon_2^2}{k}|\mathbf{A}\cdot\mathbf{e}^*|^2k^2dkd\Omega_1d\Omega_2, \quad \text{(Brems.)} \qquad (4.4)$$

where $\mathbf{A}\cdot\mathbf{e}^*$ is given by (3.3) and (3.17). Correspondingly, for pair production

$$d\sigma=\frac{1}{(2\pi)^4}\frac{e^2}{mc^2}\frac{\hbar}{mc}\frac{\epsilon_2^2}{k}|\mathbf{A}\cdot\mathbf{e}|^2p_1^2dp_1d\Omega_1d\Omega_2, \quad \text{(P.P.)} \qquad (4.5)$$

where $\mathbf{A}\cdot\mathbf{e}$ is obtained from the bremsstrahlung amplitude in the manner discussed following (3.17).

The evaluation of the absolute square of the amplitude is greatly simplified by applying (2.10), thereby eliminating two of the spinors, *viz.*,

$$\begin{aligned}|(v_2,Mv_1)|^2&=(v_1,M^\dagger v_2)(v_2,Mv_1)\\&=\tfrac{1}{2}(v_1,\,M^\dagger(1+\boldsymbol{\zeta}_2\cdot\boldsymbol{\sigma})Mv_1),\end{aligned} \qquad (4.6)$$

where $M=(\epsilon_1+\epsilon_2)\mathbf{J}\cdot\mathbf{e}^*+ik\boldsymbol{\sigma}\times\mathbf{J}\cdot\mathbf{e}^*$, and $\boldsymbol{\zeta}_2$ is the unit spin vector of the final electron. We then write $M^\dagger(1+\boldsymbol{\zeta}_2\cdot\boldsymbol{\sigma})M$ as a linear function of $\boldsymbol{\sigma}$ by repeated use of

$$\boldsymbol{\sigma}\cdot\mathbf{A}\,\boldsymbol{\sigma}\cdot\mathbf{B}=\mathbf{A}\cdot\mathbf{B}+i\boldsymbol{\sigma}\cdot\mathbf{A}\times\mathbf{B}, \qquad (4.7)$$

and finally use (2.11) to obtain $|\mathbf{A}\cdot\mathbf{e}^*|^2$ directly in terms of the unit spin vectors for the initial and final states, $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$. (This procedure is clearly to be preferred to that which would perhaps appear more straightforward, namely multiplying out the matrix element (v_2,Mv_1) explicitly using (2.8) and then taking the absolute square.) In this way one finds for bremsstrahlung

$$\begin{aligned}|\mathbf{A}\cdot\mathbf{e}^*|^2=&\tfrac{1}{2}k^2|J|^2+2\epsilon_1\epsilon_2(1+\boldsymbol{\zeta}_1\cdot\boldsymbol{\zeta}_2)|\mathbf{J}\cdot\mathbf{e}^*|^2\\&+\tfrac{1}{2}k^2\,\mathrm{Re}\{|J|^2\boldsymbol{\zeta}_1\cdot\boldsymbol{\zeta}_2-2\mathbf{J}\cdot\boldsymbol{\zeta}_1\mathbf{J}^*\cdot\boldsymbol{\zeta}_2\}\\&+k\epsilon_2\,\mathrm{Re}\{[|J|^2\boldsymbol{\zeta}_1\cdot\mathbf{e}-2\mathbf{J}\cdot\mathbf{e}\mathbf{J}^*\cdot\boldsymbol{\zeta}_1]\boldsymbol{\zeta}_2\cdot\mathbf{e}^*\}\\&-k\epsilon_1\,\mathrm{Re}\{[|J|^2\boldsymbol{\zeta}_2\cdot\mathbf{e}-2\mathbf{J}\cdot\mathbf{e}\mathbf{J}^*\cdot\boldsymbol{\zeta}_2]\boldsymbol{\zeta}_1\cdot\mathbf{e}^*\}\\&+\tfrac{1}{2}k|J|^2(\epsilon_1\boldsymbol{\zeta}_1+\epsilon_2\boldsymbol{\zeta}_2)\cdot(i\mathbf{e}\times\mathbf{e}^*)\\&+\tfrac{1}{2}k\,\mathrm{Re}\{|J|^2(\epsilon_2\boldsymbol{\zeta}_1+\epsilon_1\boldsymbol{\zeta}_2)\cdot(i\mathbf{e}\times\mathbf{e}^*)\\&-2\mathbf{J}\cdot(\epsilon_2\boldsymbol{\zeta}_1+\epsilon_1\boldsymbol{\zeta}_2)\mathbf{J}^*\cdot(i\mathbf{e}\times\mathbf{e}^*)\}\\&+\tfrac{1}{2}k[k(1+\boldsymbol{\zeta}_1\cdot\boldsymbol{\zeta}_2)(i\mathbf{e}\times\mathbf{e}^*)\\&+(\epsilon_1+\epsilon_2)(\boldsymbol{\zeta}_1\times\boldsymbol{\zeta}_2)\times(i\mathbf{e}\times\mathbf{e}^*)\\&+(\epsilon_1+\epsilon_2)(\boldsymbol{\zeta}_1+\boldsymbol{\zeta}_2)\\&-2\mathrm{Re}\{\mathbf{e}^*(\epsilon_1\boldsymbol{\zeta}_1+\epsilon_2\boldsymbol{\zeta}_2)\cdot\mathbf{e}\}]\cdot(i\mathbf{J}\times\mathbf{J}^*).\end{aligned}$$

(Brems.) (4.8)

For bremsstrahlung it is found, both in the case of no screening [BM Eq. (8.15) and ff.] and for arbitrary screening [OMW Eqs. (7b.5)-(7b.10)], that the integrals (3.5) are given by their Born-approximation values times a common factor. Thus, in the case of bremsstrahlung, $\mathbf{J}$ is real apart from a trivial phase factor, and (4.8) simplifies to

$$\begin{aligned}|\mathbf{A}\cdot\mathbf{e}^*|^2=&\tfrac{1}{2}k^2J^2+2\epsilon_1\epsilon_2(1+\boldsymbol{\zeta}_1\cdot\boldsymbol{\zeta}_2)|\mathbf{J}\cdot\mathbf{e}|^2\\&+\tfrac{1}{2}k^2[J^2\boldsymbol{\zeta}_1\cdot\boldsymbol{\zeta}_2-2\mathbf{J}\cdot\boldsymbol{\zeta}_1\mathbf{J}\cdot\boldsymbol{\zeta}_2]\\&+k\epsilon_2\,\mathrm{Re}\{[J^2\boldsymbol{\zeta}\cdot\mathbf{e}-2\mathbf{J}\cdot\mathbf{e}\mathbf{J}\cdot\boldsymbol{\zeta}_1]\boldsymbol{\zeta}_2\cdot\mathbf{e}^*\}\\&-k\epsilon_1\,\mathrm{Re}\{[J^2\boldsymbol{\zeta}_2\cdot\mathbf{e}-2\mathbf{J}\cdot\mathbf{e}\mathbf{J}\cdot\boldsymbol{\zeta}_2]\boldsymbol{\zeta}_1\cdot\mathbf{e}^*\}\\&+\tfrac{1}{2}kJ^2(\epsilon_1\boldsymbol{\zeta}_1+\epsilon_2\boldsymbol{\zeta}_2)\cdot(i\mathbf{e}\times\mathbf{e}^*)\\&+\tfrac{1}{2}k[J^2(\epsilon_2\boldsymbol{\zeta}_1+\epsilon_1\boldsymbol{\zeta}_2)\cdot(i\mathbf{e}\times\mathbf{e}^*)\\&-2\mathbf{J}\cdot(\epsilon_2\boldsymbol{\zeta}_1+\epsilon_1\boldsymbol{\zeta}_2)\mathbf{J}\cdot(i\mathbf{e}\times\mathbf{e}^*)].\end{aligned}$$

(Brems.) (4.9)

For pair production the square of the amplitude is found from (4.8) by making the substitutions $\epsilon_1\rightarrow-\epsilon_1$,

$\boldsymbol{\zeta}_1 \to -\boldsymbol{\zeta}_1$ and $\mathbf{e}^* \to \mathbf{e}$:

$$\begin{aligned}|\mathbf{A}\cdot\mathbf{e}|^2 = &\tfrac{1}{2}k^2|J|^2 - 2\epsilon_1\epsilon_2(1-\boldsymbol{\zeta}_1\cdot\boldsymbol{\zeta}_2)|\mathbf{J}\cdot\mathbf{e}^*|^2 \\ &-\tfrac{1}{2}k^2\,\mathrm{Re}\{|J|^2\boldsymbol{\zeta}_1\cdot\boldsymbol{\zeta}_2 - 2\mathbf{J}\cdot\boldsymbol{\zeta}_1\mathbf{J}^*\cdot\boldsymbol{\zeta}_2\} \\ &+k\epsilon_2\,\mathrm{Re}\{[|J|^2\boldsymbol{\zeta}_1\cdot\mathbf{e}^* - 2\mathbf{J}\cdot\mathbf{e}^*\mathbf{J}^*\cdot\boldsymbol{\zeta}_1]\boldsymbol{\zeta}_2\cdot\mathbf{e}\} \\ &+k\epsilon_1\,\mathrm{Re}\{[|J|^2\boldsymbol{\zeta}_2\cdot\mathbf{e}^* - 2\mathbf{J}\cdot\mathbf{e}^*\mathbf{J}^*\cdot\boldsymbol{\zeta}_2]\boldsymbol{\zeta}_1\cdot\mathbf{e}\} \\ &+\tfrac{1}{2}k|J|^2(\epsilon_1\boldsymbol{\zeta}_1+\epsilon_2\boldsymbol{\zeta}_2)\cdot(i\mathbf{e}\times\mathbf{e}^*) \\ &-\tfrac{1}{2}k\,\mathrm{Re}\{|J|^2(\epsilon_2\boldsymbol{\zeta}_1+\epsilon_1\boldsymbol{\zeta}_2)\cdot(i\mathbf{e}\times\mathbf{e}^*) \\ &-2\mathbf{J}\cdot(\epsilon_2\boldsymbol{\zeta}_1+\epsilon_1\boldsymbol{\zeta}_2)\mathbf{J}^*\cdot(i\mathbf{e}\times\mathbf{e}^*)\} \\ &-\tfrac{1}{2}k[k(1-\boldsymbol{\zeta}_1\cdot\boldsymbol{\zeta}_2)(i\mathbf{e}\times\mathbf{e}^*) \\ &-(\epsilon_1-\epsilon_2)(\boldsymbol{\zeta}_1\times\boldsymbol{\zeta}_2)\times(i\mathbf{e}\times\mathbf{e}^*) \\ &+(\epsilon_1-\epsilon_2)(\boldsymbol{\zeta}_1-\boldsymbol{\zeta}_2) \\ &-2\mathrm{Re}\{\mathbf{e}^*(\epsilon_1\boldsymbol{\zeta}_1+\epsilon_2\boldsymbol{\zeta}_2)\cdot\mathbf{e}\}]\cdot(i\mathbf{J}\times\mathbf{J}^*). \end{aligned} \quad \text{(P.P.)} \quad (4.10)$$

It should be noted that the cross section is symmetric with respect to the positron and electron, as it should be.

Equations (4.9) and (4.10) give the cross sections for bremsstrahlung and pair production for specified momenta of incoming and outgoing particles and for assigned directions of the initial and final spins and polarization. In the next section the cross section is integrated over the direction of motion of the final particle, giving the correlations between $\boldsymbol{\zeta}_1$, $\boldsymbol{\zeta}_2$, $\mathbf{e}$, $\mathbf{k}$ and $\mathbf{p}_1$. The spins and polarization are then successively averaged over, giving the various correlation functions.

5. CROSS SECTIONS INTEGRATED OVER DIRECTION OF MOTION OF THE FINAL PARTICLE

We shall integrate the bremsstrahlung and pair-production cross sections for an arbitrarily screened Coulomb potential over the direction of motion of the final particle, but without summing over spins. Now it has been shown[23] that if we sum over the spin of the final particle as well as integrate over its direction of motion, then the pair-production cross section may be inferred from that for bremsstrahlung by changing the sign of ϵ_2 and making the appropriate change in the final-state density factor. However, since we are using Sommerfeld-Maue type wave functions (2.1), this inference may be made even if we do not sum over the the spin of the final particle. This may be seen by noting that the difference between pair production and bremsstrahlung (apart from changes of sign in energy and momentum) results essentially from the choice of final-state wave function (i.e., whether it is of ingoing or outgoing type), which choice affects only F in (2.1). Moreover, F satisfies the spin-independent equation (2.2) and thus forms a complete set of states with the same energy but different propagation directions for *either* the outgoing or the ingoing type solution. Therefor, even if we do not sum over the spin but only integrate over the direction of motion of the final particle, the choice of final-state wave function is immaterial, i.e., the pair-production cross section may be inferred from that for bremsstrahlung.

Consider, then, pair production. As previously shown, the Coulomb and screening corrections occur for different values of the momentum transfer q, *viz.*, for large q one has only Coulomb correction, for small q only screening correction. Thus to include the effect of screening we add to the exact unscreened cross section the integrated Born-approximation screening correction, which is, apart from factors,

$$\int\{|\mathbf{A}\cdot\mathbf{e}|^2{}_{\text{Born, screened}} - |\mathbf{A}\cdot\mathbf{e}|^2{}_{\text{Born, unscreened}}\}d\Omega_2. \quad (5.1)$$

It should be noted that in this formulation the *screening* appears as a correction to the exact cross section for a pure Coulomb potential. Having used this property of the pair-production cross section to see that we need only (1) the exact integrated cross section without screening, and (2) the Born approximation screening correction, we now use the fact that these two parts of the pair-production cross section may be inferred from the corresponding bremsstrahlung cross section. This is the procedure we follow since the latter is in fact easier to evaluate.

The integrated cross section is calculated from (4.9) and (3.8), from which it may be seen that all the integrals required are of the form

$$\int \mathbf{J}\cdot\mathbf{V}_1\,\mathbf{J}\cdot\mathbf{V}_2 d\Omega_2, \quad (5.2)$$

with $\mathbf{V}_1$ and $\mathbf{V}_2$ arbitrary vectors, not depending on θ_2 or φ_2. In the next section we show that the relevant integrals are given by

$$\begin{aligned} &\int J_x^2 d\Omega_2 = X + Yu_x^2, \\ &\int J_y^2 d\Omega_2 = X + Yu_y^2, \\ &\int J_z^2 d\Omega_2 = X + Y(1-1/2\xi)^2, \\ &\int J_xJ_y d\Omega_2 = Yu_xu_y, \\ &\int J_xJ_z d\Omega_2 = Yu_x(1-1/2\xi), \\ &\int J_yJ_z d\Omega_2 = Yu_y(1-1/2\xi), \end{aligned} \quad (5.3)$$

[23] Haakon Olsen, Phys. Rev. **99**, 1335 (1955).

where, from (6.22),

$$X=(4\pi a/p_2k)^2\pi\xi^2(1+\Gamma),$$
$$Y=-(4\pi a/p_2k)^2 4\pi\xi^4\Gamma. \quad (5.3a)$$

The quantity Γ is given by (6.23) for no screening, by (6.28) and (6.29) for arbitrary screening, and by (6.34) for complete screening. Thus in the integrated expression (5.2) the terms with the factor Y will have the same form as the integrand, with the vector

$$Y\{\mathbf{u}+(1-1/2\xi)\hat{k}\}$$

replacing $\mathbf{J}$, whereas the three terms having the factor X can be written as $\mathbf{V}_1\cdot\mathbf{V}_2$. Thus we can write

$$\int \mathbf{J}\cdot\mathbf{V}_1\,\mathbf{J}\cdot\mathbf{V}_2 d\Omega_2=Y\mathbf{U}\cdot\mathbf{V}_1\,\mathbf{U}\cdot\mathbf{V}_2+X\mathbf{V}_1\cdot\mathbf{V}_2, \quad (5.4)$$

where

$$\mathbf{U}\equiv\mathbf{u}+(1-1/2\xi)\hat{k}. \quad (5.5)$$

Noting the form of the terms in (4.9) it will be convenient to observe in particular, from (5.4) and (5.3a), and noting that $U^2=(4\xi^2)^{-1}$,

$$\int J^2 d\Omega_2=\left(\frac{4\pi a}{p_2k}\right)^2\pi\xi^2(3+2\Gamma), \quad (5.6a)$$

$$\int[J^2\mathbf{V}_1\cdot\mathbf{V}_2-2\mathbf{J}\cdot\mathbf{V}_1\,\mathbf{J}\cdot\mathbf{V}_2]d\Omega_2$$
$$=(4\pi a/p_2k)^2\pi\xi^2[\mathbf{V}_1\cdot\mathbf{V}_2+8\xi^2\Gamma\mathbf{U}\cdot\mathbf{V}_1\,\mathbf{U}\cdot\mathbf{V}_2]. \quad (5.6b)$$

The integrated cross section for bremsstrahlung may now be written down from (4.4) and (4.9) using (5.4) or (5.6a, b), and $\mathbf{V}_1$ and $\mathbf{V}_2$ equal successively to $\boldsymbol{\zeta}_1$, $\boldsymbol{\zeta}_2$, $\mathbf{e}$, etc. In this way we find

$$d\sigma(\mathbf{p}_1,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2,\mathbf{k},\mathbf{e})$$
$$=Z^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2\frac{dk}{k}\frac{d\xi}{\epsilon_1^2}\frac{d\varphi_1}{2\pi}\{\tfrac{1}{2}(\epsilon_1^2+\epsilon_2^2)(3+2\Gamma)-\epsilon_1\epsilon_2$$
$$-8\epsilon_1\epsilon_2\xi^2\Gamma|\mathbf{u}\cdot\mathbf{e}|^2(1+\boldsymbol{\zeta}_1\cdot\boldsymbol{\zeta}_2)$$
$$+\tfrac{1}{2}[\epsilon_1^2+\epsilon_2^2+2\epsilon_1\epsilon_2(1+2\Gamma)]\boldsymbol{\zeta}_1\cdot\boldsymbol{\zeta}_2$$
$$+4k^2\xi^2\Gamma\boldsymbol{\zeta}_1\cdot\mathbf{U}\boldsymbol{\zeta}_2\cdot\mathbf{U}-k^2\,\mathrm{Re}\{\boldsymbol{\zeta}_1\cdot\mathbf{e}\boldsymbol{\zeta}_2\cdot\mathbf{e}^*\}$$
$$+\tfrac{1}{2}k(3+2\Gamma)(\epsilon_1\boldsymbol{\zeta}_1+\epsilon_2\boldsymbol{\zeta}_2)\cdot(i\mathbf{e}\times\mathbf{e}^*)$$
$$+\tfrac{1}{2}k(\epsilon_2\boldsymbol{\zeta}_1+\epsilon_1\boldsymbol{\zeta}_2)\cdot[(i\mathbf{e}\times\mathbf{e}^*)+8\xi^2\Gamma\mathbf{U}(\mathbf{U}\cdot(i\mathbf{e}\times\mathbf{e}^*))]$$
$$-8k\xi^2\Gamma\,\mathrm{Re}\{\mathbf{u}\cdot\mathbf{e}^*[\epsilon_1\boldsymbol{\zeta}_2\cdot\mathbf{U}\boldsymbol{\zeta}_1\cdot\mathbf{e}$$
$$-\epsilon_2\boldsymbol{\zeta}_1\cdot\mathbf{U}\boldsymbol{\zeta}_2\cdot\mathbf{e}]\}\}. \quad \text{(Brems.)} \quad (5.7)$$

Equation (5.7) is the basis for our further discussion.

6. THE INTEGRALS

For the evaluation of the integrals (5.3) we choose coordinates x, y, z with the z axis in the direction of $\mathbf{k}$, and x and y such that

$$\{u_x,u_y\}=\{u\cos\varphi_1,\,u\sin\varphi_1\},$$
$$\{v_x,v_y\}=\{v\cos\varphi_2,\,v\sin\varphi_2\}, \quad (6.1)$$

so that the integrands in (5.3) may be written in terms of φ_1 and φ_2 from (3.8). We note first that B in (3.8) depends on φ_2 only through $q_\perp^2=u^2+v^2-2uv\cos\varphi$, where $\varphi=\varphi_1-\varphi_2$. Thus it follows that

$$\int_0^{2\pi}B^2\sin(\varphi_1-\varphi_2)d\varphi_2=\int_0^{2\pi}B^2\sin 2(\varphi_1-\varphi_2)d\varphi_2=0. \quad (6.2)$$

Therefore we express the integrands in (5.3) in terms of φ_1 and φ, and, using (6.2), find indeed, directly from (3.8), that $\int J_x^2 d\Omega_2$, $\int J_y^2 d\Omega_2$, and $\int J_xJ_y d\Omega_2$ are of the form given in (5.3), with

$$X=\int B^2v^2\eta^2\sin^2\varphi d\Omega_2,$$
$$Y=\xi^2\int B^2\left(1-2\frac{v\eta}{u\xi}\cos\varphi+\frac{v^2\eta^2}{u^2\xi^2}\cos 2\varphi\right)d\Omega_2. \quad (6.3)$$

Furthermore, also from (6.2) and (3.8),

$$\int J_z^2 d\Omega_2=\int B^2(\xi-\eta)^2 d\Omega_2,$$
$$\int J_xJ_z d\Omega_2=u_x\xi\int B^2(\xi-\eta)\left(1-\frac{v\eta}{u\xi}\cos\varphi\right)d\Omega_2, \quad (6.4)$$
$$\int J_yJ_z d\Omega_2=u_y\xi\int B^2(\xi-\eta)\left(1-\frac{v\eta}{u\xi}\cos\varphi\right)d\Omega_2.$$

a. No Screening

Before proceeding further with the evaluation of the four integrals occurring in (6.3) and (6.4), we note that to include the effect of screening we have to add the Born-approximation screening correction (5.1) to the exact cross section, as discussed at the beginning of Sec. 5. We thus first calculate the integrals (6.3), (6.4) for a pure Coulomb potential, in which case [BM Eq. (8.30) and OMW Eq. (7b.11)]

$$B^2=(4\pi a/kq^2)^2R(y), \quad (6.5)$$

where

$$1-x\equiv y=\delta^2/\xi\eta q^2,$$
$$R(y)=[V^2(x)+a^2y^2W^2(x)]/V^2(1), \quad (6.6)$$

the variables appropriate to the integration being ξ, η, and y. With these new variables, we have

$$d\Omega_2=\frac{1}{2p_2^2\eta^2}\frac{\delta^2}{y^2}\frac{dyd\eta}{[-(\xi-\eta)^2+2\lambda(\xi+\eta-2\xi\eta)-\lambda^2]^{\frac{1}{2}}}, \quad (6.7)$$

where we have used the abbreviation

$$\lambda=\delta^2[(1-y)/y]. \tag{6.8}$$

Thus from (6.6) and (6.7),

$$B^2d\Omega_2=\frac{1}{2}\left(\frac{4\pi a}{p_2\delta k}\right)^2\xi^2R(y)\times\frac{dyd\eta}{[-(\xi-\eta)^2+2\lambda(\xi+\eta-2\xi\eta)-\lambda^2]^{\frac{1}{2}}}. \tag{6.9}$$

It should be noted that we may write, neglecting terms of relative order $1/\epsilon$,

$$q_\perp^2=(\delta^2/\xi\eta)[(1-y)/y], \tag{6.10}$$

which may be seen as follows: If $q=O(1)$ then $y=O(\delta^2)\ll 1$, so that $q_\perp^2\approx\delta^2/\xi\eta y=q^2$, as it should since $q_\perp\gg q_z$. For $q=O(1/\epsilon)$ we have $q_\perp^2=(\delta^2/\xi\eta y)-(\delta^2/\xi\eta)=q^2-\delta^2/\xi\eta$, which is also correct since for $q=O(1/\epsilon)$, neglecting terms of relative order $1/\epsilon$, $q_z^2=\delta^2/\xi\eta$.

In (6.3) and (6.4) we shall need $\cos\varphi$ and $\cos 2\varphi$ in terms of ξ, η, y, and to this end note that

$$2uv\xi\eta\cos\varphi=\xi+\eta-2\xi\eta-\lambda. \tag{6.11}$$

Substituting (6.9) and (6.11) in (6.3) and (6.4), we perform the η integrals first. These are

$$\int\frac{d\eta}{[-(\xi-\eta)^2+2\lambda(\xi+\eta-2\xi\eta)-\lambda^2]^{\frac{1}{2}}}=\pi,$$

$$\int\frac{\eta d\eta}{[-(\xi-\eta)^2+2\lambda(\xi+\eta-2\xi\eta)-\lambda^2]^{\frac{1}{2}}}=\pi[\xi+(1-2\xi)\lambda], \tag{6.12}$$

$$\int\frac{\eta^2d\eta}{[-(\xi-\eta)^2+2\lambda(\xi+\eta-2\xi\eta)-\lambda^2]^{\frac{1}{2}}}=\pi\{2\xi(1-\xi)\lambda(1-\lambda)+[\xi+(1-2\xi)\lambda]^2\}.$$

The limits in each integral are given by the zeros of the square root in the denominator of the integrand. From (6.9), (6.11), and (6.12) the η integrals in (6.3) and (6.4) are then

$$\int\frac{v^2\eta^2\sin^2\varphi d\eta}{[-(\xi-\eta)^2+2\lambda(\xi+\eta-2\xi\eta)-\lambda^2]^{\frac{1}{2}}}=\frac{\pi}{2}\lambda(1-\lambda),$$

$$\int\left(1-2\frac{v\eta}{u\xi}\cos\varphi+\frac{v^2\eta^2}{u^2\xi^2}\cos 2\varphi\right)\times\frac{d\eta}{[-(\xi-\eta)^2+2\lambda(\xi+\eta-2\xi\eta)-\lambda^2]^{\frac{1}{2}}}=-2\pi\lambda(1-3\lambda),$$

$$\int(\xi-\eta)^2\frac{d\eta}{[-(\xi-\eta)^2+2\lambda(\xi+\eta-2\xi\eta)-\lambda^2]^{\frac{1}{2}}}=\pi\lambda[2\xi(1-\xi)+\lambda(1-6\xi+6\xi^2)], \tag{6.13}$$

$$\int\left(1-\frac{v\eta}{u\xi}\cos\varphi\right)(\xi-\eta)\times\frac{d\eta}{[-(\xi-\eta)^2+2\lambda(\xi+\eta-2\xi\eta)-\lambda^2]^{\frac{1}{2}}}=\pi(1-2\xi)\lambda(1-3\lambda),$$

from which it follows that $\int J_z^2d\Omega_2$, $\int J_xJ_zd\Omega_2$, and $\int J_yJ_zd\Omega_2$ are also of the form given in (5.3) with X and Y as given in (6.3). To evaluate X and Y we note, from (6.9) and (6.13), that the y integrals which occur are

$$\int\lambda R(y)dy, \tag{6.14a}$$

and

$$\int\lambda^2R(y)dy. \tag{6.14b}$$

The limits on the y integral are determined by $b^2-4ac=0$ when the square root in the denominator of the η integrand is written as $(a\eta^2+b\eta+c)^{\frac{1}{2}}$, i.e., by $\lambda(1-\lambda)=0$. Thus the upper limit on y is given by $\lambda=0$, $y=1$ and the lower limit by $\lambda=1$, or, neglecting terms of relative order $1/\epsilon^2$, $y=\delta^2$. Moreover, it should be noted that since the entire range of y is covered by letting φ go from 0 to π, we must double the y integral in order to obtain the integral over the entire space Ω_2.

Thus with λ as given in (6.8) and $R(y)$ as given in (6.6), the integrand in (6.14a) is, apart from a factor δ^2,

$$\frac{x}{1-x}R(y)=\frac{1}{V^2(1)}\left[\left(\frac{V^2}{1-x}+a^2xW^2\right)-(V^2+a^2x^2W^2)\right].$$

Now from the differential equation for V, *viz.*,[24]

$$(1-x)\frac{d}{dx}\left(x\frac{dV}{dx}\right)=a^2V, \tag{6.15}$$

and from

$$dV/dx=a^2W, \tag{6.16}$$

it follows that

$$\frac{d}{dx}(xVW)=\frac{V^2}{1-x}+a^2xW^2, \tag{6.17}$$

and

$$\frac{d}{dx}[xV^2-x^2(1-x)a^2W^2]=V^2+a^2x^2W^2, \tag{6.18}$$

[24] Davies, Bethe, and Maximon, Phys. Rev. **93**, 788 (1954); in the following referred to as DBM. Note DBM, Eqs. (30)–(32), and reference 14, Eqs. (8.12)–(8.14).

from which

$$\int_0^{1-\delta^2}\frac{x}{1-x}R(y)dx$$

$$=\frac{1}{V^2(1)}[xVW-xV^2+x^2(1-x)a^2W^2]\Big|_0^{1-\delta^2} \quad (6.19)$$

$$=-\ln\delta^2-1-2f(Z)+O(\delta^2\ln\delta),$$

where we have used, from DBM Eq. (34),

$$W(1-y)=-V(1)[\ln y+2f(Z)]+O(y\ln y) \quad \text{for } y\ll 1, \quad (6.20)$$

$$f(Z)=a^2\sum_{n=1}^{\infty}\frac{1}{n(n^2+a^2)}.$$

In (6.14b) the contribution to the integral coming from $y=O(1)$ will, because of the extra factor δ^2 in λ, be of $O(\delta^2)$ relative to the entire integral. Hence the only nonnegligible contribution to (6.14b) comes from $y=O(\delta^2)$, in which case we can write $R(y)=1$, $\lambda=\delta^2/y$, and

$$\int_{\delta^2}^1 \lambda^2 R(y)dy=\delta^2+O(\delta^4\ln\delta). \quad (6.21)$$

Thus, substituting (6.9) in (6.3), using the η integrations (6.13) and the y integrations (6.19) and (6.21) [which must be doubled according to the remark following (6.14b)], we have

$$X=(4\pi a/p_2k)^2\pi\xi^2(1+\Gamma),$$
$$Y=-(4\pi a/p_2k)^2 4\pi\xi^4\Gamma, \quad (6.22)$$

where

$$\Gamma=\ln(1/\delta)-2-f(Z). \quad (6.23)$$

b. Screening

As we have noted, the effect of screening is included by adding to the exact unscreened cross section the Born-approximation screening correction, (5.1). As in the case of the exact cross section, the Born-approximation differential cross section is given by (4.9), with **J** as given in (3.8). However, the factor B in (3.8) is, in the Born approximation including screening, given by

$$B=\frac{4\pi a}{k}\frac{[1-F(q)]}{q^2} \quad (6.24)$$

rather than by (6.5). $F(q)$ is the atom form factor.[22] Since $F(q)$ is only given numerically, the variables appropriate to the integration are now ξ, η, and q rather than ξ, η, and y. However, we are here concerned only with the screening *correction*, which is significant for $q\lesssim Z^{\frac{1}{3}}/137\ll 1$. This permits us to use the variables ξ, η, y of the unscreened case since $|\xi-\eta|<q$ and hence for these small q we may replace η by ξ in the argument $q=\delta/(\xi\eta y)^{\frac{1}{2}}$ of $F(q)$, which then becomes $F(\delta/\xi\sqrt{y})$, independent of η. The η integrals are then as before and are given in (6.13). Therefore, since it follows that merely after integrating over η (but not over y) the integrals shown in (5.3) have the form given there, the Born-approximation screening correction to these integrals, with

$$B^2d\Omega_2=\frac{1}{2}\left(\frac{4\pi a}{p_2\delta k}\right)^2\xi^2\frac{\{[1-F(\delta/\xi\sqrt{y})]^2-1\}d\eta dy}{[-(\xi-\eta)^2+2\lambda(\xi+\eta-2\xi\eta)-\lambda^2]^{\frac{1}{2}}}, \quad (6.25)$$

will also be of the form (5.3) with X and Y given by (6.3) and $B^2d\Omega_2$ as in (6.25). In (6.3) the integral over η is given in (6.13), but these expressions are, in the present region $q\ll 1$, much simpler since $\lambda<\delta^2/y\leqq q^2\ll 1$, and hence we need only retain the terms of first order in λ. The y integral occurring in both X and Y is, therefore,

$$\int\{[1-F(\delta/\xi\sqrt{y})]^2-1\}\lambda dy.$$

As in (6.14a, b), the upper limit, which corresponds to small q, is $y=1$. The lower limit may be taken to be $y=\delta^2$ rather than $y=\delta^2/q_0^2$ ($r_{\text{sc}}^{-1}\ll q_0\ll 1$, where r_{sc} is the screening radius), since for $y<\delta^2/q_0^2$ the integrand is zero. Changing now to the variable $q\equiv\delta/\epsilon\sqrt{y}$, we have

$$\int_{\delta^2}^1\{[1-F(\delta/\xi\sqrt{y})]^2-1\}\lambda dy$$
$$=2\delta^2\int_{\delta/\xi}^{\infty}\{[1-F(q)]^2-1\}\frac{(q^2-\delta^2/\xi^2)}{q^3}dq. \quad (6.26)$$

The upper limit in the q integral in (6.26) is taken to be infinite but may be chosen to be any value of $q\geqslant q_0$. The X and Y to be substituted in (5.3) to obtain the Born-approximation screening correction to the integrals in (5.3) are therefore, from (6.3), (6.13), (6.25), and (6.26),

$$X_{\text{sc. corr}}=(4\pi a/p_2k)^2\pi\xi^2\mathfrak{F}(\delta/\xi),$$
$$Y_{\text{sc. corr}}=-(4\pi a/p_2k)^2 4\pi\xi^4\mathfrak{F}(\delta/\xi), \quad (6.27)$$

where

$$\mathfrak{F}(\delta/\xi)=\int_{\delta/\xi}^{\infty}\{[1-F(q)]^2-1\}\frac{(q^2-\delta^2/\xi^2)}{q^3}dq. \quad (6.28)$$

Therefore the X and Y for arbitrary screening are again given by (6.22), but where now

$$\Gamma=\ln(1/\delta)-2-f(Z)+\mathfrak{F}(\delta/\xi). \quad (6.29)$$

This, then, is the general expression for Γ to be used in (5.7). Γ is always positive for the high energies considered here.

We have calculated $\mathfrak{F}(\delta/\xi)$ for the Thomas-Fermi

TABLE I. $\mathfrak{F}(\delta/\xi)$ for the Thomas-Fermi model of screening as used by Molière.

$Z^{1/3}6\xi/121\delta-\mathfrak{F}(\delta/\xi)$	0.5	1.0	2.0	4.0	8.0	15.0	20.0		
	0.0144	0.0492	0.1400	0.3312	0.6758	1.127	1.367		
$Z^{1/3}6\xi/121\delta-\mathfrak{F}(\delta/\xi)$	25.0	30.0	40.0	50.0	60.0	70.0	80.0	90.0	100.0
	1.564	1.731	2.001	2.216	2.393	2.545	2.676	2.793	2.897

model as used by Molière, *viz.*,[25]

$$\frac{1-F(q)}{q^2}=\sum_{i=1}^{3}\frac{\alpha_i}{\beta_i^2+q^2}, \tag{6.30}$$

with

$$\alpha_1=0.10,\quad \alpha_2=0.55,\quad \alpha_3=0.35,$$

$$\beta_i=(Z^{1/3}/121)b_i;\quad b_1=6.0,\quad b_2=1.20,\quad b_3=0.30.$$

The result is rather simple:

$$\mathfrak{F}(\delta/\xi)=-\tfrac{1}{2}\sum_{i=1}^{3}\alpha_i^2\ln(1+B_i) +\sum_{\substack{i=1\\ i\neq j}}^{3}\sum_{j=1}^{3}\alpha_i\alpha_j\left[\frac{1+B_j}{B_i-B_j}\ln(1+B_j)+\tfrac{1}{2}\right]. \tag{6.31}$$

We use the abbreviation

$$B_i=(\beta_i\xi/\delta)^2. \tag{6.32}$$

In the special case of complete screening, $\beta_i\xi/\delta\gg 1$, we get, from (6.31),

$$\mathfrak{F}(\delta/\xi)=\ln(111Z^{-1/3}\delta/\xi), \tag{6.33}$$

giving

$$\Gamma=\ln(111Z^{-1/3}/\xi)-2-f(Z).\quad \text{(Compl. Sc.)} \tag{6.34}$$

For the Thomas-Fermi model of screening as used by Molière, (6.30), $\mathfrak{F}(\delta/\xi)$ is given in Table I.

7. THE POLARIZATION OF BREMSSTRAHLUNG

From (5.7) we may write down the polarization of the emitted radiation and the depolarization of the electron. In Secs. 7 and 8 we consider the polarization of the radiation, leaving the depolarization to Sec. 9.

To avoid complicated expressions we shall consider only correlations between the sets of variables $(\mathbf{p}_1,\boldsymbol{\zeta}_1,\mathbf{k},\mathbf{e})$ and $(\mathbf{p}_1,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2,\mathbf{e})$. The first set gives the angular dependence of the polarization of the emitted radiation and is discussed in this section. The latter contains the correlation between the three polarization variables $\boldsymbol{\zeta}_1$, $\boldsymbol{\zeta}_2$, and $\mathbf{e}$; this cross section, discussed in Sec. 8, is considered only after the angular dependence of the radiation has been integrated out. When the final electron is not observed, the cross section is

$$\begin{aligned} d\sigma(\mathbf{p}_1,\boldsymbol{\zeta}_1,\mathbf{k},\mathbf{e}) &=Z^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2\frac{dk}{k}\frac{d\xi}{\epsilon_1^2}\frac{d\varphi_1}{2\pi}\{(\epsilon_1^2+\epsilon_2^2)(3+2\Gamma)\\ &\quad-2\epsilon_1\epsilon_2(1+4u^2\xi^2\Gamma)-8\epsilon_1\epsilon_2u^2\xi^2\Gamma(2|\hat{u}\cdot\mathbf{e}|^2-1)\\ &\quad+[(\epsilon_1^2-\epsilon_2^2)(3+2\Gamma)-2k\epsilon_2(1+4u^2\xi^2\Gamma)]\boldsymbol{\zeta}_1\cdot\hat{k}\\ &\quad\times(i\mathbf{e}\times\mathbf{e}^*)\cdot\hat{k}-4k\epsilon_2\xi(1-2\xi)\\ &\quad\times\Gamma\boldsymbol{\zeta}_1\cdot\mathbf{u}(i\mathbf{e}\times\mathbf{e}^*)\cdot\hat{k}\}. \end{aligned} \tag{7.1}$$

This result was given in a previous note.[12] The radiation is seen to be elliptically polarized, the major axis of the ellipse being perpendicular to the plane of emission, since the coefficient of $(2|\hat{u}\cdot\mathbf{e}|^2-1)$ is always negative. The linear polarization of the radiation is independent of the polarization of the initial electron when the final electron spin is not observed. The cross section summed over polarization directions, i.e., the angular distribution of the radiation, is

$$\begin{aligned} d\sigma(\mathbf{p}_1,\mathbf{k})&=2Z^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2\frac{dk}{k}\frac{d\xi}{\epsilon_1^2}\\ &\quad\times\{(\epsilon_1^2+\epsilon_2^2)(3+2\Gamma)-2\epsilon_1\epsilon_2(1+4u^2\xi^2\Gamma)\}. \end{aligned} \tag{7.2}$$

As the distribution is independent of φ_1, we have integrated over this variable.

The linear polarization is given by

$$P=\frac{d\sigma_\perp-d\sigma_\parallel}{d\sigma_\perp+d\sigma_\parallel},$$

where $d\sigma_\perp$ and $d\sigma_\parallel$ are the cross sections for bremsstrahlung polarized perpendicular and parallel to the emission plane, respectively.

$$P(\mathbf{p}_1,\mathbf{k},\mathbf{e}_{\text{linear}})=\frac{8\epsilon_1\epsilon_2u^2\xi^2\Gamma}{(\epsilon_1^2+\epsilon_2^2)(3+2\Gamma)-2\epsilon_1\epsilon_2(1+4u^2\xi^2\Gamma)}. \tag{7.3}$$

This expression is an extension of the results of May[2] and Gluckstern *et al.*[3] in that it takes into account the screening exactly and includes the Coulomb correction. The radiation is linearly polarized perpendicular to the plane of emission. The maximum polarization for any element in the case of no screening occurs for $\xi=\frac{1}{2}$, or $u=p_1\theta_1=1$. Screening will increase this value of u

[25] G. Molière, Z. Naturforsch. **2a**, 133 (1947).

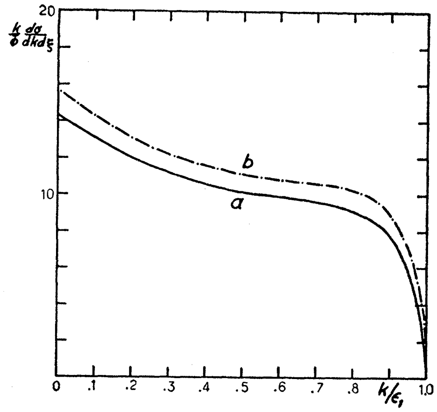

FIG. 1. Bremsstrahlung spectrum, $d\sigma(\mathbf{p}_1,\mathbf{k})/dkd\xi$, of 50-Mev electrons in lead at a photon emission angle $\theta_1=\epsilon_1^{-1}=10^{-2}$ rad. $\bar{\phi}=Z^2(e^2/\hbar c)(e^2/mc^2)^2$. Curve a and b as explained in Fig. 2.

slightly. The polarization is always maximum at the lower end of the spectrum, while it is smallest at the upper end.

The circular polarization, $P=d\sigma_r-d\sigma_l/d\sigma_r+d\sigma_l$, where $d\sigma_r$ and $d\sigma_l$ are the cross sections for right and left circularly polarized bremsstrahlung[19] respectively, behaves in exactly the opposite way, increasing with increasing k. In general, the circular polarization from longitudinally polarized electrons is considerably greater than from transversely polarized electrons. Note that because of the small angle between $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{k}$, $\boldsymbol{\zeta}_1\cdot\hat{k}=\boldsymbol{\zeta}_1\cdot\hat{p}_1$ when one neglects terms of relative order $1/\epsilon$.

The circular polarization from completely longitudinally polarized electrons, $\boldsymbol{\zeta}_1\cdot\hat{p}_1=\pm 1$, is given by

$$P(\mathbf{p}_1,\boldsymbol{\zeta}_{1\,\mathrm{long}},\mathbf{k},\mathbf{e}_{\mathrm{circ}})=\pm\frac{k[(\epsilon_1+\epsilon_2)(3+2\Gamma)-2\epsilon_2(1+4u^2\xi^2\Gamma)]}{(\epsilon_1^2+\epsilon_2^2)(3+2\Gamma)-2\epsilon_1\epsilon_2(1+4u^2\xi^2\Gamma)}. \quad (7.4)$$

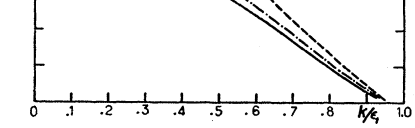

FIG. 2. Linear polarization of bremsstrahlung, $P(\mathbf{p}_1,\mathbf{k},\mathbf{e}_{\mathrm{linear}})$, perpendicular to the plane of emission, of 50-Mev electrons in lead at a photon emission angle $\theta_1=\epsilon_1^{-1}=10^{-2}$ rad. Curve a: Exact calculation involving Coulomb correction and screening. Curve b: Born-approximation calculation with only screening taken into account. Curve c: Born-approximation calculation neglecting screening.

It is remarkable that at the upper end of the spectrum, $k\approx\epsilon_1$, the polarization of the radiation is complete for any element.

The circular polarization of bremsstrahlung from completely transversely polarized electrons, $\boldsymbol{\zeta}_1\cdot\hat{p}_1=0$, is

$$P(\mathbf{p}_1,\boldsymbol{\zeta}_{1\,\mathrm{trans}},\mathbf{k},\mathbf{e}_{\mathrm{circ}})=-\frac{(\boldsymbol{\zeta}_1\cdot\mathbf{u})4k\epsilon_2\xi(1-2\xi)\Gamma}{(\epsilon_1^2+\epsilon_2^2)(3+2\Gamma)-2\epsilon_1\epsilon_2(1+4u^2\xi^2\Gamma)}. \quad (7.5)$$

This contribution is zero for just those values for which (7.4) is maximum, *viz.*, $\xi=\frac{1}{2}$ and $k=\epsilon_1$. Only for values far from these is (7.5) comparable to (7.4). It should also be noted that $P(\mathbf{p}_1,\boldsymbol{\zeta}_{1\,\mathrm{trans}},\mathbf{k},\mathbf{e}_{\mathrm{circ}})$, Eq. (7.5), is maximum when $\boldsymbol{\zeta}_1$ is in the plane of emission, while it is zero when the photon is emitted in a plane perpendicular to $\boldsymbol{\zeta}_1$.

In Figs. 1–3 we give curves for the cross section and the linear and circular polarizations for $\xi=\frac{1}{2}$ and $\epsilon_1=100$. In this case the circular polarization from transversely polarized electrons, $P(\mathbf{p}_1,\boldsymbol{\zeta}_{1\,\mathrm{trans}},\mathbf{k},\mathbf{e}_{\mathrm{circ}})$, is zero. This quantity is exhibited in Fig. 4 for $\epsilon_1=100$ for the case when it is close to its maximum value, namely for $u=0.414$. $P(\mathbf{p}_1,\boldsymbol{\zeta}_{1\,\mathrm{trans}},\mathbf{k},\mathbf{e}_{\mathrm{circ}})$ is much smaller than $P(\mathbf{p}_1,\boldsymbol{\zeta}_{1\,\mathrm{long}},\mathbf{k},\mathbf{e}_{\mathrm{circ}})$. As shown by the curves, the screening and Coulomb corrections are quite important for the spectrum and for the linear polarization. However, these corrections do not have any significant influence on the circular polarization.

8. THE POLARIZATION CORRELATION

In Secs. 8 and 9 we consider the dependence of the cross section on the polarization variables $\boldsymbol{\zeta}_1$, $\boldsymbol{\zeta}_2$, and $\mathbf{e}$. Since at the high energies considered here the momenta $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{k}$ are all inside a very narrow cone of opening angle of order $1/\epsilon$, we may still give a meaning to the polarization of the photon and electron beams (Secs. 8 and 9, respectively), determined from the cross section integrated over the direction of motion of the photon as well as that of the final electron.

a. No Screening

The cross section for final spin $\boldsymbol{\zeta}_2$ and polarization of radiation $\mathbf{e}$, when the initial electron has spin $\boldsymbol{\zeta}_1$, obtained by integrating (5.7) over the direction of motion of $\mathbf{k}$ (φ_1 and ξ), is particularly simple for the case of no screening:

$$\begin{aligned}d\sigma(\mathbf{p}_1,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2,\mathbf{e})&=\tfrac{1}{2}Z^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2\frac{dk}{\epsilon_1^2k}(3+2\Gamma)\{\epsilon_1^2+\epsilon_2^2-\tfrac{2}{3}\epsilon_1\epsilon_2\\&\quad+\tfrac{1}{3}(\epsilon_1+\epsilon_2)^2\boldsymbol{\zeta}_1\cdot\boldsymbol{\zeta}_2-\tfrac{2}{3}k^2\,\mathrm{Re}\{\boldsymbol{\zeta}_1\cdot\mathbf{e}\,\boldsymbol{\zeta}_2\cdot\mathbf{e}^*\}\\&\quad+k[(\epsilon_1+\tfrac{1}{3}\epsilon_2)\boldsymbol{\zeta}_1+(\epsilon_2+\tfrac{1}{3}\epsilon_1)\boldsymbol{\zeta}_2]\cdot(i\mathbf{e}\times\mathbf{e}^*)\}.\end{aligned} \quad (8.1)$$

Since $3+2\Gamma$ appears as a common factor, the various polarizations are in this case independent of Γ and therefore independent of the Coulomb correction $f(Z)$. Thus the Born-approximation values for the polarizations of the beam are exact in the present case. The linear part of the polarization of the radiation is perpendicular to the line dividing the angle between $\boldsymbol{\zeta}_{1\perp}$ and $\boldsymbol{\zeta}_{2\perp}$ into two equal parts. It should be noted that this linear polarization is proportional to k^2; thus it is zero at the lower end of the spectrum. The linear polarization is

$$P(\mathbf{p}_1,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2,\mathbf{e}_{\text{linear}})=\frac{k^2|\boldsymbol{\zeta}_{1\perp}||\boldsymbol{\zeta}_{2\perp}|}{(\epsilon_1^2+\epsilon_2^2)(3+\zeta_{1z}\zeta_{2z})-2\epsilon_1\epsilon_2(1-2\boldsymbol{\zeta}_{1\perp}\cdot\boldsymbol{\zeta}_{2\perp}-\zeta_{1z}\zeta_{2z})}. \quad (8.2)$$

It is maximum when the spins are both completely transverse, $\zeta_{1z}=\zeta_{1z}=0$, and antiparallel, $\boldsymbol{\zeta}_{1\perp}=-\boldsymbol{\zeta}_{2\perp}$. At this maximum, P is the same all over the spectrum:

$$P(\mathbf{p}_1,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2,\mathbf{e}_{\text{linear}})=\tfrac{1}{3}.$$

The circular polarization of the photon beam is

$$P(\mathbf{p}_1,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2,\mathbf{e}_{\text{circ}})=\frac{3k[(\epsilon_1+\frac{1}{3}\epsilon_2)\zeta_{1z}+(\epsilon_2+\frac{1}{3}\epsilon_1)\zeta_{2z}]}{(\epsilon_1^2+\epsilon_2^2)(3+\zeta_{1z}\zeta_{2z})-2\epsilon_1\epsilon_2(1-2\boldsymbol{\zeta}_{1\perp}\cdot\boldsymbol{\zeta}_{2\perp}-\zeta_{1z}\zeta_{2z})}. \quad (8.3)$$

The elliptic photon polarization is given by

$$P(\mathbf{p}_1,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2,\mathbf{e}_{\text{ell}})=[P^2(\mathbf{e}_{\text{linear}})+P^2(\mathbf{e}_{\text{circ}})]^{\frac{1}{2}}. \quad (8.4)$$

b. Arbitrary Screening

When integrating (5.7) for arbitrary screening over φ_1 and ξ we get, instead of (8.1),

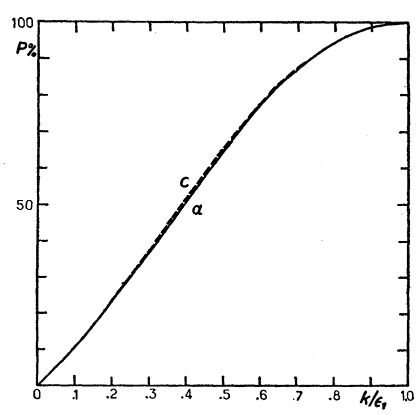

FIG. 3. Circular polarization of bremsstrahlung of 50-Mev electrons in lead, polarized in the direction of motion, $P(\mathbf{p}_1,\boldsymbol{\zeta}_{1\,\text{long}},\mathbf{k},\mathbf{e}_{\text{circ}})$, at a photon emission angle $\theta_1=\epsilon_1^{-1}=10^{-2}$ rad. Curves a and c as in Fig. 2.

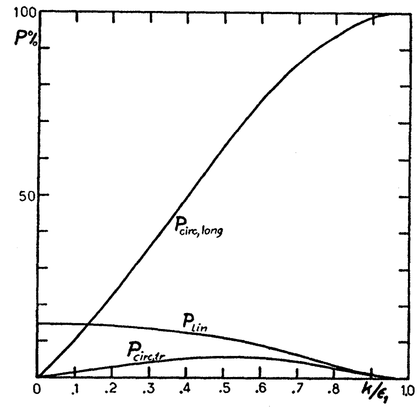

FIG. 4. Linear polarization of bremsstrahlung,

$$P_{\text{lin}}=P(\mathbf{p}_1,\mathbf{k},\mathbf{e}_{\text{linear}}),$$

circular polarization of bremsstrahlung from longitudinally polarized electrons, $P_{\text{circ long}}=P(\mathbf{p}_1,\boldsymbol{\zeta}_{1\,\text{long}},\mathbf{k},\mathbf{e}_{\text{circ}})$, and from transversely polarized electrons,

$$P_{\text{circ trans}}=P(\mathbf{p}_1,\boldsymbol{\zeta}_{1\,\text{trans}},\mathbf{k},\mathbf{e}_{\text{circ}})/(\boldsymbol{\zeta}_1\cdot\hat{u}),$$

of 50-Mev electrons in lead at a photon emission angle

$$\theta_1=0.41\epsilon_1^{-1}=0.41\times10^{-2}\text{ rad.}$$

Coulomb and screening effects are included.

$$d\sigma(\mathbf{p}_1,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2,\mathbf{e})=\tfrac{1}{4}Z^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2\frac{dk}{\epsilon_1^2k}\{(\epsilon_1^2+\epsilon_2^2)\psi_1-\tfrac{2}{3}\epsilon_1\epsilon_2\psi_2$$
$$+(\epsilon_1^2+\epsilon_2^2)(\psi_1-\tfrac{2}{3}\psi_2)\zeta_{1z}\zeta_{2z}$$
$$+2\epsilon_1\epsilon_2[(\psi_1-\tfrac{1}{3}\psi_2)\boldsymbol{\zeta}_{1\perp}\cdot\boldsymbol{\zeta}_{2\perp}+\tfrac{1}{3}\psi_2\zeta_{1z}\zeta_{2z}]$$
$$+\tfrac{1}{3}k^2\psi_2(\boldsymbol{\zeta}_{1\perp}\cdot\boldsymbol{\zeta}_{2\perp}-2\,\mathrm{Re}\{\boldsymbol{\zeta}_1\cdot\mathbf{e}\;\boldsymbol{\zeta}_2\cdot\mathbf{e}^*\})$$
$$+k\psi_1(\epsilon_1\boldsymbol{\zeta}_1+\epsilon_2\boldsymbol{\zeta}_2)\cdot(i\mathbf{e}\times\mathbf{e}^*)$$
$$+k(\psi_1-\tfrac{2}{3}\psi_2)(\epsilon_2\boldsymbol{\zeta}_1+\epsilon_1\boldsymbol{\zeta}_2)\cdot(i\mathbf{e}\times\mathbf{e}^*)\}. \quad (8.5)$$

ψ_1 and ψ_2 are given by[26]

$$\psi_1=6+4\int_0^1\Gamma(\xi)d\xi,$$
$$\psi_2=6+24\int_0^1\xi(1-\xi)\Gamma(\xi)d\xi. \quad (8.6)$$

[26] The integrals (8.6) for the case of arbitrary screening are most easily evaluated by substituting $\Gamma(\xi)$ as given by (6.29) and (6.28) [choosing the upper limit in (6.28) to be 1 rather than ∞], omitting the region $0<\xi<\delta$ in (8.6) (the lower limit then being δ), changing to the variable $x=\delta/\xi$, and integrating successively by parts. One then obtains

$$\psi_1=4\left[\int_\delta^1(q-\delta)^2(1-F(q))^2q^{-3}dq+1-f(Z)\right]$$

$$\psi_2=4\left[\int_\delta^1(q^3-6\delta^2q\ln(q/\delta)+3\delta^2q-4\delta^3)(1-F(q))^2\times q^{-4}dq+\tfrac{5}{6}-f(Z)\right].$$

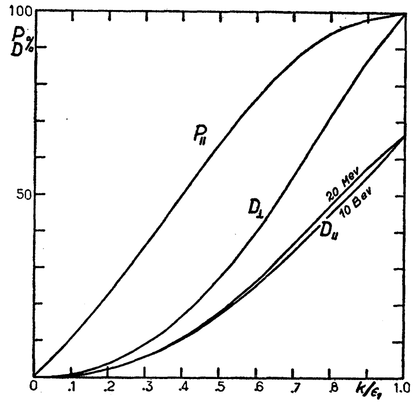

FIG. 5. Circular polarization of bremsstrahlung beam from longitudinally polarized electrons,

$$P_{\parallel}=P(\mathbf{p}_1,\boldsymbol{\zeta}_{1\,\text{long}},\mathbf{e}_{\text{circ}}),$$

and depolarization of longitudinally polarized electrons,

$$D_{\parallel}=D(\mathbf{p}_1,\boldsymbol{\zeta}_{1\,\text{long}})$$

and of transversely polarized electrons, $D_\perp=D(\mathbf{p}_1,\boldsymbol{\zeta}_{1\,\text{trans}})$. Coulomb and screening effects are included. The curves for $P_\parallel$ and $D_\perp$ are valid for all elements and for any incident electron energy above ≈ 20 Mev. $D_\parallel$ depends slightly on the electron energy; curves are shown for incident electron energies 20 Mev and 10 Bev.

From the Bethe-Heitler spectrum for arbitrary screening, it follows that [26,27]

$$\begin{aligned}\psi_1&=\phi_1-\tfrac{4}{3}\ln Z-4f(Z),\\ \psi_2&=\phi_2-\tfrac{4}{3}\ln Z-4f(Z).\end{aligned}\tag{8.7}$$

ϕ_1 and ϕ_2 are the functions given[27] and tabulated[28] by Bethe and Heitler. It is remarkable that also in the polarization-dependent parts of the cross section only these functions appear.

The expressions for the polarizations for arbitrary screening analogous to Eqs. (8.2) and (8.3) are given below.

The linear polarization of the photon beam is

$$P(\mathbf{p}_1,\zeta_1,\zeta_2,\mathbf{e}_{\text{linear}})=k^2\psi_2|\boldsymbol{\zeta}_{1\perp}||\boldsymbol{\zeta}_{2\perp}|/\mathfrak{N}\tag{8.8}$$

where

$$\begin{aligned}\mathfrak{N}=(\epsilon_1^2+\epsilon_2^2)[3\psi_1+(3\psi_1-2\psi_2)\zeta_{1z}\zeta_{2z}]\\ -2\epsilon_1\epsilon_2[\psi_2-(3\psi_1-2\psi_2)\boldsymbol{\zeta}_{1\perp}\cdot\boldsymbol{\zeta}_{2\perp}-\psi_2\zeta_{1z}\zeta_{2z}].\end{aligned}$$

At the lower end of the spectrum, because of the factor k^2, the polarization is very small and hence the screening correction does not visibly affect the polarization curve. At the upper end of the spectrum, on the other hand, there is no screening correction at all, since $\delta>q_0$ $[F(q)=0$ for $q\geqq q_0]$ for large k. Thus in general the screening correction to the polarization curve will not be very important. This is also true for the other quantities in this section.

The circular polarization is given by

$$\begin{aligned}P(\mathbf{p}_1,\zeta_1,\zeta_2,\mathbf{e}_{\text{circ}})=3k[(\epsilon_1\zeta_{1z}+\epsilon_2\zeta_{2z})\psi_1\\ +(\epsilon_2\zeta_{1z}+\epsilon_1\zeta_{2z})(\psi_1-\tfrac{2}{3}\psi_2)]/\mathfrak{N},\end{aligned}\tag{8.9}$$

where $\mathfrak{N}$ is given in (8.8).

Of greatest interest is perhaps the circular polarization of the photon beam irrespective of the spin of the final electron:

$$P(\mathbf{p}_1,\zeta_1,\mathbf{e}_{\text{circ}})=\frac{k\zeta_{1z}[\epsilon_1\psi_1+\epsilon_2(\psi_1-\frac{2}{3}\psi_2)]}{(\epsilon_1^2+\epsilon_2^2)\psi_1-\frac{2}{3}\epsilon_1\epsilon_2\psi_2}.\tag{8.10}$$

The quantity $P(\mathbf{p}_1,\zeta_1,\mathbf{e}_{\text{circ}})$ is shown in Fig. 5. Screening has no influence on the curve; thus an excellent approximation to (8.10) for all energies is the expression for no screening,

$$P(\mathbf{p}_1,\zeta_1,\mathbf{e}_{\text{circ}})=\frac{k(\epsilon_1+\frac{1}{3}\epsilon_2)\zeta_{1z}}{\epsilon_1^2+\epsilon_2^2-\frac{2}{3}\epsilon_1\epsilon_2}.\tag{8.11}$$

As this quantity depends only upon the ratio k/ϵ_1, the curve in Fig. 5 is valid for any value of ϵ_1.

In order to check the Molière representation for the Thomas-Fermi model, Eq. (6.30), we have calculated ϕ_1 and ϕ_2 from the integrated expressions (8.6) and from (8.7):

$$\phi_1=6+4\int_0^1\Gamma(\xi)d\xi+\tfrac{4}{3}\ln Z+4f(Z),$$

or, introducing the expression (6.29) for $\Gamma(\xi)$,

$$\phi_1=19.25-4\ln\gamma+4\int_0^1\mathfrak{F}(\delta/\xi)d\xi.\tag{8.12}$$

γ is the quantity defined in Eq. (61) of reference 27:

$$\gamma=100kZ^{-\frac{1}{3}}/\epsilon_1\epsilon_2.\tag{8.13}$$

In the same way we find

$$\phi_2=19.25-4\ln\gamma+24\int_0^1\xi(1-\xi)\mathfrak{F}(\delta/\xi)d\xi.\tag{8.14}$$

Using the expression (6.31) for $\mathfrak{F}(\delta/\xi)$, we find

$$\begin{aligned}\int_0^1\mathfrak{F}(\delta/\xi)d\xi=-\tfrac{1}{2}\sum_{i=1}^3\alpha_iA(\beta_i/\delta)\\ +\sum_{\substack{i=1\\i\neq j}}^3\sum_{j=1}^3\alpha_i\alpha_j\frac{\beta_i^2}{\beta_j^2-\beta_i^2}A(\beta_i/\delta)\\ +\sum_{\substack{i=1\\i\neq j}}^3\sum_{j=1}^3\frac{\delta^2}{\beta_z^2-\beta_i^2}B(\beta_i/\delta)\\ +\tfrac{1}{2}\sum_{\substack{i=1\\i\neq j}}^3\sum_{j=1}^3\alpha_i\alpha_j.\end{aligned}\tag{8.15}$$

Here

$$A(\beta_i/\delta)=\ln[1+(\beta_i/\delta)^2]+2(\delta/\beta_i)\tan^{-1}(\beta_i/\delta)-2,$$

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[27] H. A. Bethe, Proc. Cambridge Phil. Soc. **30**, 524 (1934).

[28] H. A. Bethe and W. Heitler, Proc. Roy. Soc. (London) **A146**, 83 (1934).

TABLE II. Comparison of values of ϕ_1 computed from Eqs. (8.12)–(8.16) with values taken from curve of Bethe and Heitler.[a]

γ	0	0.4	0.826	1.22	1.65	2.00
ϕ_1 (Eq. (8.9)	20.84	19.28	18.01	17.08	16.23	15.68
ϕ_1(Bethe-Heitler)	20.84	19.28	18.00	17.08	16.22	15.64

[a] See reference 28.

and

$$B(\beta_i/\delta)=-\ln[1+(\beta_i/\delta)^2]+2(\beta_i/\delta)\tan^{-1}(\beta_i/\delta)-2. \tag{8.16}$$

In terms of Bethe's variable γ, $\beta_i/\delta=1.653b_i/\gamma$.

We have computed ϕ_1 from Eqs. (8.12)–(8.16). The result is compared in Table II with the values taken from the curve of Bethe and Heitler.[28] The excellent agreement shows that the Molière representation (6.30) is satisfactory.

9. DEPOLARIZATION

It will be useful to denote by $d\sigma(\mathbf{p}_1,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2)$ the cross section integrated over the direction of motion of the final electron and the photon, and summed over the polarizations of the photon. Further, we denote by $d\sigma_{\text{flip}}$ and $d\sigma_{\text{no flip}}$ the value of $d\sigma(\mathbf{p}_1,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2)$ with $\boldsymbol{\zeta}_2=-\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2=\boldsymbol{\zeta}_1$, respectively, and let $d\sigma=d\sigma_{\text{flip}}+d\sigma_{\text{no flip}}$ be $d\sigma(\mathbf{p}_1,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2)$ summed over final spins.

The depolarization of the electron because of bremsstrahlung is then

$$D(\mathbf{p}_1,\boldsymbol{\zeta}_1)=1-\left[\frac{d\sigma_{\text{no flip}}-d\sigma_{\text{flip}}}{d\sigma_{\text{no flip}}+d\sigma_{\text{flip}}}\right]=2d\sigma_{\text{flip}}/d\sigma. \tag{9.1}$$

For the case of no screening we have, from (8.1),

$$d\sigma_{\text{flip}}=Z^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2\frac{kdk}{\epsilon_1^2}(3+2\Gamma)(1-\tfrac{1}{3}\zeta_{1z}^2), \qquad \text{(No sc.)} \tag{9.2}$$

and

$$D(\mathbf{p}_1\boldsymbol{\zeta}_1)=\frac{k^2(1-\frac{1}{3}\zeta_{1z}^2)}{\epsilon_1^2+\epsilon_2^2-\frac{2}{3}\epsilon_1\epsilon_2}. \qquad \text{(No sc.)} \tag{9.3}$$

From (9.3) the depolarization is only important for the harder quanta, the upper part of the spectrum. The depolarization of a transversely polarized electron, $\zeta_{1z}=0$, is $\frac{2}{3}$ times the depolarization of a longitudinally polarized electron, $\zeta_{1z}=1$.

We may also derive the mean depolarization per centimeter path length, τ_{dep}, when the electron passes through a material with density of atoms N:

$$\tau_{\text{dep}}=N\int D(\mathbf{p}_1,\boldsymbol{\zeta}_1)\frac{d\sigma}{dk}dk=2N\sigma_{\text{flip}}, \tag{9.4}$$

where σ_{flip} is the total cross section for spin flip:

$$\sigma_{\text{flip}}=\int_0^{\epsilon_1-1}\frac{d\sigma_{\text{flip}}}{dk}dk. \tag{9.5}$$

Thus for no screening we have, from (8.1) and (6.23),

$$\tau_{\text{dep}}=2NZ^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2(1-\tfrac{1}{3}\zeta_{1z}^2)\times[\ln(2\epsilon_1)-\tfrac{3}{2}-f(Z)]. \qquad \text{(No sc.)} \tag{9.6}$$

Analogously to the radiation length L_{rad}, defined by $L_{\text{rad}}=\tau_{\text{rad}}^{-1}$ where

$$\tau_{\text{rad}}=N\int_0^{\epsilon_1-1}\frac{k}{\epsilon_1}\frac{d\sigma}{dk}dk \tag{9.7}$$

is the mean energy loss per centimeter path length, we define the depolarization length $L_{\text{dep}}=\tau_{\text{dep}}^{-1}$. For no screening we have, from (8.1),

$$L_{\text{rad}}^{-1}=\tau_{\text{rad}}=4NZ^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2[\ln(2\epsilon_1)-\tfrac{1}{3}-f(Z)]. \qquad \text{(No sc.)} \tag{9.8}$$

Thus there is a close relationship between these two lengths, *viz.*, for the case of no screening,

$$\frac{L_{\text{dep}}}{L_{\text{rad}}}=\frac{2[\ln(2\epsilon_1)-\frac{1}{3}-f(Z)]}{(1-\frac{1}{3}\zeta_{1z}^2)[\ln(2\epsilon_1)-\frac{3}{2}-f(Z)]}. \qquad \text{(No sc.)} \tag{9.9}$$

In general, for arbitrary screening, the spin-flip part of the cross section is, from (8.5),

$$d\sigma_{\text{flip}}=\tfrac{1}{2}Z^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2\frac{kdk}{\epsilon_1^2}\{\psi_1-\zeta_{1z}^2(\psi_1-\tfrac{2}{3}\psi_2)\}, \qquad \text{(Arb. sc.)} \tag{9.10}$$

the depolarization of the electron spin is

$$D(\mathbf{p}_1,\boldsymbol{\zeta}_1)=\frac{k^2[\psi_1-\zeta_{1z}^2(\psi_1-\frac{2}{3}\psi_2)]}{(\epsilon_1^2+\epsilon_2^2)\psi_1-\frac{2}{3}\epsilon_1\epsilon_2\psi_2}, \qquad \text{(Arb. sc.)} \tag{9.11}$$

and the mean depolarization per centimeter path length is

$$\tau_{\text{dep}}=\tfrac{1}{2}NZ^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2\frac{1}{\epsilon_1^2}\int_0^{\epsilon_1-1}k[\psi_1-\zeta_{1z}^2(\psi_1-\tfrac{2}{3}\psi_2)]dk. \qquad \text{(Arb. sc.)} \tag{9.12}$$

At extremely high energies, in which case the screening is complete, we find, using (6.34) and (8.6),

$$\psi_1=4\ln(111Z^{-\frac{1}{3}})+2-4f(Z)=4[\ln(183Z^{-\frac{1}{3}})-f(Z)],$$
$$\psi_2=4[\ln(183Z^{-\frac{1}{3}})-f(Z)]-\tfrac{2}{3}. \qquad \text{(Compl. sc.)} \tag{9.13}$$

Thus for complete screening the spin-flip part of the cross section is

$$d\sigma_{\mathrm{flip}}=2Z^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2\frac{kdk}{\epsilon_1^2}\times\{[\ln(183Z^{-\frac{1}{3}})-f(Z)][1-\tfrac{1}{3}\zeta_{1z}^2]-\tfrac{1}{9}\}\qquad\text{(Compl. sc.)}\quad(9.14)$$

and the cross section summed over final spins is

$$d\sigma=4Z^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2\frac{dk}{\epsilon_1^2 k}\{(\epsilon_1^2+\epsilon_2^2-\tfrac{2}{3}\epsilon_1\epsilon_2)\times[\ln(183Z^{-\frac{1}{3}})-f(Z)]+\tfrac{1}{9}\epsilon_1\epsilon_2\}.\qquad\text{(Compl. sc.)}\quad(9.15)$$

Substituting (9.14) and (9.15) in (9.7) and (9.4), we have

$$\frac{L_{\mathrm{dep}}}{L_{\mathrm{rad}}}=\frac{2[\ln(183Z^{-\frac{1}{3}})-f(Z)+1/18]}{[\ln(183Z^{-\frac{1}{3}})-f(Z)][1-\frac{1}{3}\zeta_{1z}^2]-\frac{1}{9}}.\qquad\text{(Compl. sc.)}\quad(9.16)$$

If in (9.16) we neglect the small quantities 1/9 and 1/18, we obtain the very simple relation

$$L_{\mathrm{dep}}=\frac{2}{1-\frac{1}{3}\zeta_{1z}^2}L_{\mathrm{rad}}.\qquad(9.17)$$

For the case of no screening, if we neglect the numbers $\frac{1}{3}$ and $\frac{3}{2}$, we find the same expression, (9.17), for L_{dep}. Therefore this simple relation between the depolarization and radiation lengths is always approximately valid.

10. PAIR PRODUCTION

From the formulas in Secs. 7 and 8 we may obtain the corresponding formulas for pair production by the substitutions $\epsilon_2,\boldsymbol{\zeta}_2\rightarrow-\epsilon_2,-\boldsymbol{\zeta}_2$ while all the other variables, ϵ_1, $\mathbf{p}_1$, $\boldsymbol{\zeta}_1$ and k, $\mathbf{k}$, $\mathbf{e}$ remain unchanged. In connection with the change from outgoing to ingoing waves one should consult Sec. 5 where this question is discussed in some detail. In the cross section the statistical factor must be changed, $k^2dk\rightarrow p_1^2dp_1$, as is well known.

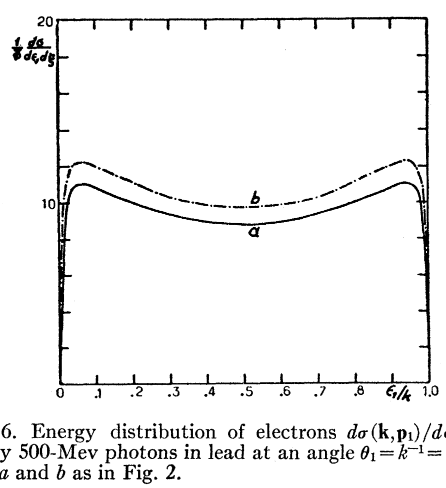

FIG. 6. Energy distribution of electrons $d\sigma(\mathbf{k},\mathbf{p}_1)/d\epsilon_1 d\xi$ produced by 500-Mev photons in lead at an angle $\theta_1=k^{-1}=10^{-3}$ rad. Curves a and b as in Fig. 2.

It should be noted that the alternative substitutions ϵ_1, $\mathbf{p}_1$, $\boldsymbol{\zeta}_1\rightarrow-\epsilon_1$, $-\mathbf{p}_1$, $-\boldsymbol{\zeta}_1$ and k, $\mathbf{k}$, $\mathbf{e}\rightarrow-k$, $-\mathbf{k}$, $\mathbf{e}^*$, while leaving ϵ_2, $\boldsymbol{\zeta}_2$ unchanged, leads to the same result as the one used above. This follows since the processes described by these matrix elements are inverse processes. The change in statistical factor is the same as above.

Also in this way we obtain from (5.7) the cross section for pair production by a quantum $\mathbf{k}$ with polarization $\mathbf{e}$ when the direction of motion of one of the particles is integrated out:

$$\begin{aligned}d\sigma(\mathbf{k},\mathbf{e},\mathbf{p}_1,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2)&=Z^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2\frac{d\epsilon_1}{k^3}d\xi\frac{d\varphi_1}{2\pi}\{\tfrac{1}{2}(\epsilon_1^2+\epsilon_2^2)(3+2\Gamma)\\&+\epsilon_1\epsilon_2+8\epsilon_1\epsilon_2\xi^2\Gamma|\mathbf{u}\cdot\mathbf{e}|^2(1-\boldsymbol{\zeta}_1\cdot\boldsymbol{\zeta}_2)\\&-\tfrac{1}{2}[\epsilon_1^2+\epsilon_2^2-2\epsilon_1\epsilon_2(1+2\Gamma)]\boldsymbol{\zeta}_1\cdot\boldsymbol{\zeta}_2\\&-4k^2\xi^2\Gamma\boldsymbol{\zeta}_1\cdot\mathbf{U}\boldsymbol{\zeta}_2\cdot\mathbf{U}+k^2\,\mathrm{Re}\{\boldsymbol{\zeta}_1\cdot\mathbf{e}\boldsymbol{\zeta}_2\cdot\mathbf{e}^*\}\\&+\tfrac{1}{2}k(3+2\Gamma)(\epsilon_1\boldsymbol{\zeta}_1+\epsilon_2\boldsymbol{\zeta}_2)\cdot(i\mathbf{e}\times\mathbf{e}^*)\\&-\tfrac{1}{2}k(\epsilon_2\boldsymbol{\zeta}_1+\epsilon_1\boldsymbol{\zeta}_2)\cdot[(i\mathbf{e}\times\mathbf{e}^*)+8\xi^2\Gamma\mathbf{U}(\mathbf{U}\cdot(i\mathbf{e}\times\mathbf{e}^*))]\\&+8k\xi^2\Gamma\,\mathrm{Re}\{\mathbf{u}\cdot\mathbf{e}^*[\epsilon_1\boldsymbol{\zeta}_2\cdot\mathbf{U}\boldsymbol{\zeta}_1\cdot\mathbf{e}+\epsilon_2\boldsymbol{\zeta}_1\cdot\mathbf{U}\boldsymbol{\zeta}_2\cdot\mathbf{e}]\}\}.\end{aligned}\qquad(10.1)$$

Here

$$\mathbf{U}=\mathbf{u}+(1-1/2\xi)\hat{k},\qquad(10.2)$$

which is the same as in (5.5). Γ is given by the same expressions as in the case of bremsstrahlung, Eqs. (6.23), (6.28)–(6.29), and (6.34) for the cases of no screening, arbitrary screening, and complete screening, respectively.

Equation (10.1) may also be derived directly from (4.10) and (3.20), performing the integrals as was done in Secs. 5 and 6 in the case of bremsstrahlung. The required integrals are similar to (5.2), of the form $\int\mathbf{J}\cdot\mathbf{V}_1\mathbf{J}^*\cdot\mathbf{V}_2d\Omega_2$. For the case of no screening, with $\mathbf{J}$ as given in (3.20), the η integrals which occur are given in (6.12) and the integrands in the y integrals are the exact differentials which appear on the right-hand side of Eqs. (6.16) and (6.17). y and x are given by (3.20a) and $\lambda=\xi\eta q_\perp^2=y-\delta^2$. The result of these integrations (although considerably more tedious to perform than in the case of bremsstrahlung) is identical to (5.4) given for bremsstrahlung, *viz.*: $\int\mathbf{J}\cdot\mathbf{V}_1\mathbf{J}^*\cdot\mathbf{V}_2d\Omega_2=Y\mathbf{U}\cdot\mathbf{V}_1\mathbf{U}\cdot\mathbf{V}_2+X\mathbf{V}_1\cdot\mathbf{V}_2$ where X and Y are given in (5.3a), with the obvious modification for pair production, $k=\epsilon_1+\epsilon_2$. In particular it follows that $\int\mathbf{J}\times\mathbf{J}^*d\Omega_2=0$, and hence that

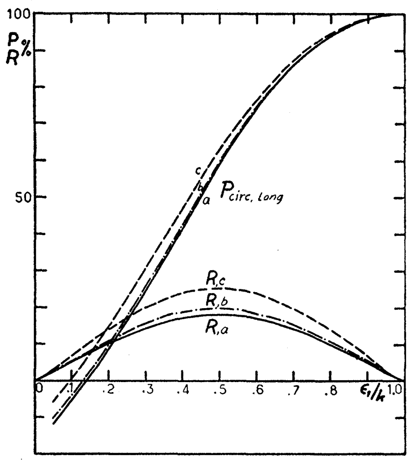

FIG. 7. Asymmetry ratio, $R(\mathbf{k},\mathbf{e}_{\text{linear}},\mathbf{p}_1)$, of electrons produced by linearly polarized 500-Mev photons in lead at an electron-photon angle $\theta_1=k^{-1}=10^{-3}$ rad, and longitudinal polarization of electron produced in lead by a circularly polarized photon of the same energy and at the same angle. Curves a, b and c as explained in Fig. 2.

the integral of the last four terms in (4.10) over Ω_2 is zero.

As is the case of bremsstrahlung, we now alternatively sum over $\boldsymbol{\zeta}_2$ and integrate over the directions of $\mathbf{p}_1$. In the former case we obtain the cross section for the production of an electron (or positron) with momentum $\mathbf{p}_1$ and spin $\boldsymbol{\zeta}_1$; in the latter case the spin correlation and other polarization properties of the electron-positron beam as a whole are obtained.

The cross section for production of an electron with momentum $\mathbf{p}_1$ and spin $\boldsymbol{\zeta}_1$ is, from (7.1),

$$\begin{aligned} d\sigma(\mathbf{k},\mathbf{e},\mathbf{p}_1,\boldsymbol{\zeta}_1) &= Z^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2\frac{d\epsilon_1}{k^3}d\xi\frac{d\varphi_1}{2\pi}\{(\epsilon_1^2+\epsilon_2^2)(3+2\Gamma) \\ &\quad +2\epsilon_1\epsilon_2(1+4u^2\xi^2\Gamma)+8\epsilon_1\epsilon_2u^2\xi^2\Gamma(2|\hat{u}\cdot\mathbf{e}|^2-1) \\ &\quad +[(\epsilon_1^2-\epsilon_2^2)(3+2\Gamma)+2k\epsilon_2(1+4u^2\xi^2\Gamma)] \\ &\quad \times\boldsymbol{\zeta}_1\cdot\hat{k}(i\mathbf{e}\times\mathbf{e}^*)\cdot\hat{k} \\ &\quad +4k\epsilon_2\xi(1-2\xi)\Gamma\boldsymbol{\zeta}_1\cdot\mathbf{u}(i\mathbf{e}\times\mathbf{e}^*)\cdot\hat{k}\}. \qquad (10.3)\end{aligned}$$

The cross section for pair production by unpolarized photons when the spin is not observed is

$$\begin{aligned} d\sigma(\mathbf{k},\mathbf{p}_1) &= 2Z^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2\frac{d\epsilon_1}{k^3}d\xi\{(\epsilon_1^2+\epsilon_2^2)(3+2\Gamma) \\ &\quad +2\epsilon_1\epsilon_2(1+4u^2\xi^2\Gamma)\}. \qquad (10.4)\end{aligned}$$

The energy distribution of electrons, $d\sigma(\mathbf{k},\mathbf{p}_1)/d\epsilon_1 d\xi$, produced by 500-Mev photons in lead at an angle $\theta_1=k^{-1}=10^{-3}$ rad, is shown in Fig. 6.

When the photon is linearly polarized, the electron is most likely to be emitted *in* the plane of polarization, the asymmetry ratio then being

$$\begin{aligned} R &= \frac{d\sigma(\hat{u}=\mathbf{e})-d\sigma(\hat{u}\cdot\mathbf{e}=0)}{d\sigma(\hat{u}=\mathbf{e})+d\sigma(\hat{u}\cdot\mathbf{e}=0)} \\ &= \frac{8\epsilon_1\epsilon_2u^2\xi^2\Gamma}{(\epsilon_1^2+\epsilon_2^2)(3+2\Gamma)+2\epsilon_1\epsilon_2(1+4u^2\xi^2\Gamma)}. \qquad (10.5)\end{aligned}$$

R is greatest when the electron and positron are equally fast, and is minimum when either one of them is very slow. Even at the maximum it is never very large, at most of the order of 20%.

A circularly polarized photon will produce a polarized electron-positron pair. From (10.3) we find the longitudinal polarization

$$\begin{aligned} P(\mathbf{k},\mathbf{e}_{\text{circ}},\mathbf{p}_1,\boldsymbol{\zeta}_{1\text{ long}}) &= \frac{\pm k[(\epsilon_1-\epsilon_2)(3+2\Gamma)+2\epsilon_2(1+4u^2\xi^2\Gamma)]}{(\epsilon_1^2+\epsilon_2^2)(3+2\Gamma)+2\epsilon_1\epsilon_2(1+4u^2\xi^2\Gamma)}; \qquad (10.6)\end{aligned}$$

$+$ and $-$ stand for right- and left-handed polarized photon[19], respectively. The faster one of the pair particles is always polarized to a high degree in the same sense as the circularly polarized photon; at the upper end of the energy distribution, $\epsilon_1\approx k$, $\epsilon_2\approx 1$, the polarization is 100% for any element. The slower one of the particles is polarized in the opposite sense to that of the photon at the lower end of the energy spectrum. This is also shown in Figs. 7 and 8 for the case of $k=1000$.

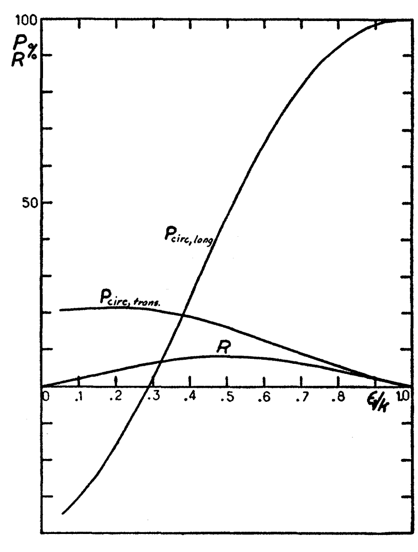

FIG. 8. Asymmetry ratio, $R(\mathbf{k},\mathbf{e}_{\text{linear}},\mathbf{p}_1)$, of electrons produced by linearly polarized 500-Mev photons in lead at an electron-photon angle $\theta_1=0.41k^{-1}$, and longitudinal polarization, $P_{\text{circ long}}=P(\mathbf{k},\mathbf{e}_{\text{circ}},\mathbf{p}_1,\boldsymbol{\zeta}_{1\text{ long}})$, and transversal polarization,

$$P_{\text{circ trans}}=P(\mathbf{k},\mathbf{e}_{\text{circ}},\mathbf{p}_1,\boldsymbol{\zeta}_{1\text{ trans}}),$$

of electrons produced by circularly polarized photons in lead at the same angle and energy, Coulomb and screening effects are included.

The screening and Coulomb corrections are seen to be relatively unimportant for the polarizations.

The amount of transverse polarization of the spin is in general smaller than the amount of longitudinal polarization. From (10.3) we find the transverse polarization

$$P(\mathbf{k},\mathbf{e}_{\text{circ}},\mathbf{p}_1,\boldsymbol{\zeta}_{1\text{ trans}}) = \frac{\pm 4k\epsilon_2 u\xi(1-2\xi)\Gamma}{(\epsilon_1^2+\epsilon_2^2)(3+2\Gamma)+2\epsilon_1\epsilon_2(1+4u^2\xi^2\Gamma)}. \quad (10.7)$$

Only when the longitudinal polarization passes through zero (Fig. 8) is the transverse polarization of importance; thus only in a limited range of energies is one of the pair particles transversely polarized.

When the cross section (10.1) is integrated over $d\Omega_1$ but not summed over $\boldsymbol{\zeta}_2$, then the spin polarization correlation of the electron-positron beam is obtained:

$$\begin{aligned} d\sigma(\mathbf{k},\mathbf{e},\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2) &= \tfrac{1}{4}Z^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2\frac{d\epsilon_1}{k^3}\{(\epsilon_1^2+\epsilon_2^2)\psi_1+\tfrac{2}{3}\epsilon_1\epsilon_2\psi_2 \\ &\quad -(\epsilon_1^2+\epsilon_2^2)(\psi_1-\tfrac{2}{3}\psi_2)\zeta_{1z}\zeta_{2z} \\ &\quad +2\epsilon_1\epsilon_2[(\psi_1-\tfrac{1}{3}\psi_2)\boldsymbol{\zeta}_{1\perp}\cdot\boldsymbol{\zeta}_{2\perp}+\tfrac{1}{3}\psi_2\zeta_{1z}\zeta_{2z}] \\ &\quad -\tfrac{1}{3}k^2\psi_2(\boldsymbol{\zeta}_{1\perp}\cdot\boldsymbol{\zeta}_{2\perp}-2\,\mathrm{Re}\{\boldsymbol{\zeta}_1\cdot\mathbf{e}\boldsymbol{\zeta}_2\cdot\mathbf{e}^*\}) \\ &\quad +k\psi_1(\epsilon_1\boldsymbol{\zeta}_1+\epsilon_2\boldsymbol{\zeta}_2)\cdot(i\mathbf{e}\times\mathbf{e}^*) \\ &\quad -k(\psi_1-\tfrac{2}{3}\psi_2)(\epsilon_2\boldsymbol{\zeta}_1+\epsilon_1\boldsymbol{\zeta}_2)\cdot(i\mathbf{e}\times\mathbf{e}^*)\}. \quad (10.8)\end{aligned}$$

Here ψ_1 and ψ_2 are the same functions as occur in the case of bremsstrahlung; they are given in (8.6). After averaging over polarization and summing over spins, we are left with the Bethe-Heitler terms including Coulomb correction:

$$d\sigma(\mathbf{k},\mathbf{p}_1) = Z^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2\frac{d\epsilon_1}{k^3} \times\{(\epsilon_1^2+\epsilon_2^2)\psi_1+\tfrac{2}{3}\epsilon_1\epsilon_2\psi_2\}. \quad (10.9)$$

The spin correlation of the electron positron pair produced by an unpolarized photon is

$$C = \frac{d\sigma(\boldsymbol{\zeta}_1=\boldsymbol{\zeta}_2)-d\sigma(\boldsymbol{\zeta}_1=-\boldsymbol{\zeta}_2)}{d\sigma(\boldsymbol{\zeta}_1=\boldsymbol{\zeta}_2)+d\sigma(\boldsymbol{\zeta}_1=-\boldsymbol{\zeta}_2)} = \frac{2\epsilon_1\epsilon_2(\psi_1-\tfrac{1}{3}\psi_2)-k^2(\psi_1-\tfrac{2}{3}\psi_2)\zeta_{1z}^2}{(\epsilon_1^2+\epsilon_2^2)\psi_1+\tfrac{2}{3}\epsilon_1\epsilon_2\psi_2}. \quad (10.10)$$

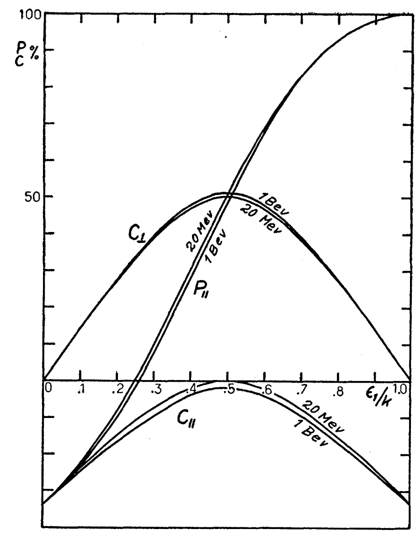

FIG. 9. Longitudinal polarization of the electron beam, $P_\parallel = P(\mathbf{k},\mathbf{e}_{\text{circ}},\boldsymbol{\zeta}_1)$, the spin correlation of longitudinal spins, $C_\parallel = C(\mathbf{k},\boldsymbol{\zeta}_{1\text{ long}},\boldsymbol{\zeta}_{2\text{ long}})$, and of transverse spins,

$$C_\perp = C(\mathbf{k},\boldsymbol{\zeta}_{1\text{ trans}},\boldsymbol{\zeta}_{2\text{ trans}}),$$

of electron-positron pairs. Coulomb and screening effects are included. The dependence on the photon energy k is very small. Curves are shown for incident photon energies 20 Mev and 1 Bev.

The longitudinal polarization of the electron beam produced by a circularly polarized photon is

$$P(\mathbf{k},\mathbf{e}_{\text{circ}},\boldsymbol{\zeta}_1) = \frac{\pm k[\epsilon_1\psi_1-\epsilon_2(\psi_1-\tfrac{2}{3}\psi_2)]}{(\epsilon_1^2+\epsilon_2^2)\psi_1+\tfrac{2}{3}\epsilon_1\epsilon_2\psi_2}. \quad (10.11)$$

The transverse polarization of the electron beam is zero, as it should be.

The quantities P and C depend only slightly on the photon energy. This is shown in Fig. 9 where curves for incident photon energies 20 Mev and 1 Bev are given.

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